1. (a) (5 points) Write down the existence and uniqueness theorem for initial value problems.

(b) (5 points) Show that the ODE $\frac{dy}{dx} = (y - 2)^{\frac{1}{3}}$ with initial value $y(0) = 2$ has two solutions.

(c) (5 points) What part of the existence/uniqueness theorem fails to hold in the above example?
2. (15 points) Mice in my house follow the population law \( \frac{dx}{dt} = (\alpha - \beta)x - \gamma x^2 \), where \( \alpha, \beta, \gamma \) are positive parameters. Explain using a bifurcation diagram why there are no mice if \( \alpha < \beta \) and a stable population of mice if \( \alpha > \beta \).

Find an exact value for the equilibrium population for \( \alpha > \beta \).
3. The second order ODE $\varepsilon y'' = y' - (x^3 - x)$ is to be solved by a singular perturbation.

   (a) (5 points) Set $u = y$ and $v = y'$. Graph the approximate solution with $\varepsilon = 0$ in the phase plane.

   (b) (5 points) In the above picture, include a sketch what an actual solution with initial condition $y(0) = 0$, $y'(0) = 1$ would look like.

   (c) (5 points) Explain why the actual solutions such as the one above must converge rapidly to the approximate solution.
4. A spring-mass system satisfies \( m \frac{d^2h}{dt^2} + b \frac{dh}{dt} + h = f. \) \( h \) represents height (length), and \( t \) is time, \( m \) is mass, \( f \) is force.

(a) (5 points) Let \( L \) be the length of the spring and use \( x = h/L \) to get a new differential equation for \( x \).

(b) (5 points) Rescale time and divide by whatever is in front of the \( h \) term to get a dimensionless equation.

(c) (5 points) Find a condition in terms of dimensionless parameters such that the highest derivative term can be neglected. On what time scale \( T \) does your approximation hold?
5. (a) (10 points) For \( x' = r(10 - x) - sx^r x \) show that for some small \( r \) there are two stable positions, but if \( r \) is large there is only one.

(b) (5 points) What happens if \( x \) starts at the lower critical point when \( r \) is small, but \( r \) is raised gradually to some high value? Then what happens if \( r \) is lowered back to the original value?
6. (15 points) Draw the bifurcation diagram for $x' = rx - \frac{x}{1+x^2}$. What is this bifurcation called?
7. (15 points) Runner $A$ follows runner $B$ around a circular track. If $\phi$ is the difference in their positions then $\phi$ follows the equation $\phi' = \gamma - K \cos(\phi)$, where $K$ is a fixed positive constant. Show that if $\gamma$ is large then $A$ will always be falling behind $B$, but if $\gamma$ is small then $A$ will settle to a fixed position behind $B$. What is the latter situation called?