172A HOMEWORK SOLUTIONS

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1. Chapter 1

1.4.4

\[ 1000(1 + st) = 1700 \]
\[ 700 = 8000s \]
\[ s = 0.0875 \]

1.4.5

\[ 1200(1 + sT) = 1320 \]
\[ 1 + sT = 1.1 \Rightarrow \]
\[ sT = 0.1 \]

Note that we don’t need to figure out what \( s \) and \( T \) are individually- the final answer is \( 500(1 + 2sT) = 500(1 + 0.2) = 600. \)

1.4.6  

a) The problem should really read "During what one year period is the interest rate 1/23." Then,

\[ \frac{1}{23} = \frac{(1 + ((n + 1)/20)) - (1 + n/20)}{1 + n/20} \]
\[ 1 + n/20 = 23 + 23(n + 1)/20 - 23 - 23n/20 \]
\[ 20 + n = 23 \]
\[ n = 3. \]

So the interval is [3, 4].

b) 

\[ i_{[1,6]} = \frac{(1 + (0.05)(6)) - (1 + 0.05(4))}{1 + 0.05(4)} \]
\[ = (1.3 - 1.2)/1.2 \]
\[ = 8.33\% \]
1.5.1

\[2200(1.04)^t = 8000\]
\[t \ln 1.05 = \ln \frac{8000}{2200} \Rightarrow\]
\[t = 32.92\]

1.5.2

\[P(1.062)^{13} = 32168 \Rightarrow\]
\[P = 14716.53\]

1.5.6

\[A(10) = 826(1.03)^3(1.04)^2(1.05)^5\]
\[= 1245.96\]

1.5.7

\[(1 + i)^{14} = (1.05)^8(1.006)^72\]
\[= 2.2728\]
\[i = 2.2728^{1/14} - 1\]
\[= 6.04\%\]

1.6.5 Start by solving for the effective rate, \(i\).

\[0.2 = \frac{a(4.5) - a(2)}{a(2)}\]
\[= \frac{a(2)(1 + i)^{2.5} - a(2)}{a(2)}\]
\[= (1 + i)^{2.5} - 1 \Rightarrow\]
\[i = 7.565\%\]

We can now solve for \(d_{1.3}\).

\[d_{1.3} = \frac{a(3) - a(1)}{a(3)}\]
\[= \frac{a(1)(1 + i)^2 - a(1)}{a(1)(1 + i)^2}\]
\[= \frac{(1 + i)^2 - 1}{(1 + i)^2}\]
\[= 13.57\%\]
1.7.4

\[ 6000 = X(1.065)^{-2} + 2X(1.065)^{-1} \]
\[ = 0.88166X + 1.5546X \]
\[ = 2.4363X \]
\[ X = 2462.74 \]

1.7.5

\[ 5000(1.04)^{-3}(1.05)^{-2}(1.055)^{-5} = 3084.81 \]

1.7.8 Solve for \( X \).

\[ -20000 + 8000(1.06)^{-1} + 15000(1.06)^{-2} = -10000 - X(1.06)^{-2} + 3000(1.06)^{-1} + 14000(1.06)^{-3} \]
\[ -20000 + 7547.17 + 13349.95 = -10000 - 0.89X + 2830.19 + 11754.67 \]
\[ 897.12 = 4584.86 - 0.89X \]
\[ X = 4143.03 \]

Then,

\[ NPV(1.5\%) = -20000 + 8000(1.05)^{-1} + 15000(1.05)^{-2} \]
\[ = 1224.49 \]

\[ NPV(2.5\%) = -10000 - 4143.03(1.05)^{-2} + 3000(1.05)^{-1} + 14000(1.05)^{-3} \]
\[ = -10000 - 3758.32 + 2857.14 + 12093.73 \]
\[ = 1,192.55. \]

The difference is therefore 1224.49 - 1192.55 = 31.94.

1.9.3 First calculate \( d \).

\[ d = \frac{i}{i + 1} = \frac{0.068}{1.068} \]
\[ = 6.367\% \]
Now calculate $d^{(4)}$.

\[
d^{(4)} = 4 \left( 1 - (1 - d)^{1/4} \right) \\
= 4 \left( 1 - (1 - 0.06367)^{1/4} \right) \\
= 6.525\% \Rightarrow \\
\frac{d^{(4)}}{4} = 1.6312\%
\]

1.10.1

\[
\left( 1 - \frac{d^{(4)}}{4} \right)^{-4} = (1 - d)^{-1} = 1 + i \\
i = (1 - 0.02)^{-4} - 1 = 8.4166\% \\
i^{(6)} = 6 \left( (1 + i)^{1/6} - 1 \right) \\
= 6 \left( 1.084166^{1/6} - 1 \right) \\
= 8.1358\% \\
d = \frac{i}{i + 1} = \frac{0.084166}{1.084166} \\
= 7.7632\% \\
d^{(3)} = 3 \left( 1 - (1 - d)^{1/3} \right) \\
= 3 \left( 1 - (0.922368)^{1/3} \right) \\
= 7.9732\%
\]

1.10.4

\[
2480 \left( 1 + \frac{0.02}{12} \right)^{36} \left( 1 - \frac{0.03}{2} \right)^{-4} \left( 1 + \frac{0.042}{1/2} \right)^2 (1 - 0.058)^{-3} = \\
2480(1.06178)(1.06232)(1.17506)(1.19632) = 3932.32
\]

1.11.2

\[
300e^{0.04(5)} = 366.42
\]

1.11.3 Convert all the rates to annual effective rates.

\[
i_A = 5.2\% \\
i_B = 1.0044^{12} - 1 = 5.4097\% \\
i_C = e^{0.0516} - 1 = 5.2954\%
\]
Account A would have the lowest accumulation.

R.1

\[
6208 \left(1 - \frac{0.023}{4}\right)^{12} \left(1 + \frac{0.03}{12}\right)^{12} (1 - 0.042)^{-3} e^{(0.040)(2)} = \\
6208(1.047213)(1.030416)(1.137374)(1.096365) = 8353.29
\]

R.2 Here's an outline. Let \( p = 0.00107584 = d^{(2)} - d \). Using this condition, show that \( d \) must satisfy the quadratic equation

\[
d^2 + 2pd + p^2 - 4p = 0
\]

Solve this quadratic for \( d \) (check: \( d = 6.452\% \)). From here, calculate \( i \) and \( i^{(3)} \) directly.

R.5 Equate the NPV's.

\[
6000 + 4000(1.05)^{-1} = 12000(1.05)^{-N/12}
\]

\[
1.05^{-N/12} = 0.81746
\]

\[
\ln 0.81746 = \ln 1.05
\]

\[
N = 49.572
\]

2. Chapter 2

2.2.1

\[
K(1-d)^{-3} = 982
\]

\[
K = 982(1-0.04)^3 = 868.81
\]

2.2.3 First solve for the effective annual rate:

\[
i = \left(1 + \frac{j^{(12)}}{12}\right)^{12} - 1
\]

\[
= \left(1 + \frac{0.032}{12}\right)^{12} - 1
\]

\[
\approx 3.2167\%
\]

\[
d^2 + 2(d^{(2)} - d) d + (d^{(2)} - d)^2 - 4(d^{(2)} - d) = 0
\]

\[
\frac{[d^{(2)}]^2}{c_1} - d^{(2)} = -d
\]

\[
1 - d^{(2)} + \frac{[d^{(2)}]^2}{c_1} = 1 - d
\]

\[
\left(1 - \frac{d^{(2)}}{2}\right)^2 = (1 - d) \iff \sqrt{\}
\]
Now we can solve for the time:

\[ 1800(1 + i)^t = 1965.35 \Rightarrow \]
\[ t \ln 1.032467 = \ln \frac{1965.35}{1800} \Rightarrow \]
\[ t = 2.7503 \]

2.3.1 Set the NPV equal to zero and solve.

\[ 12000 = 4000v + Xv^2 + 3000v^3 \]
\[ = 4000(1 - d) + X(1 - d)^2 + 3000(1 - d)^3 \]
\[ = 4000(0.94) + X(0.94)^2 + 3000(0.94)^3 \]
\[ = 3760 + 0.8836X + 2491.75 \Rightarrow \]
\[ X = \frac{(12000 - 3760 - 2491.75)}{0.8836} = 6505.49 \]

2.3.3 The NPV of the cash flows should be zero. This implies the equation:

\[ 12(1.06)^{-2T} + 10(1.06)^{-T} - 20 = 0 \]

Let \( x = 1.06^{-T} \) so that we have a quadratic equation:

\[ 12x^2 + 10x - 20 = 0 \Rightarrow \]
\[ x = \frac{-10 \pm \sqrt{100 - 4(12)(-20)}}{24} \]
\[ \approx \frac{-10 \pm 32.56}{24} \]
\[ \approx -10 \pm 1.36 \]

We only care about the positive solution, which simplifies to about 0.9399. Now solve for \( T \):

\[ 1.06^{-T} = x = 0.9399 \Rightarrow T \approx 1.0637. \]

2.3.4 The equation we’d like to solve is,

\[ 3500(1 + i) + 500(1 + i)^{1/2} + 800(1 + i)^{1/4} = 5012 \]
This equation is difficult to solve algebraically (in particular, it is not quadratic). We must resort to the CF function. We enter:

\[ CF0 = 3500 \]
\[ C1 = 0 \]
\[ F1 = 1 \]
\[ C2 = 500 \]
\[ F2 = 1 \]
\[ C3 = 800 \]
\[ F3 = 1 \]
\[ C4 = -5012 \]
\[ F4 = 1 \]

\( CPT, IRR \)

This gives the result 1.317933. What this means is the quarterly effective rate is 1.317933%. To get the annual effective rate,

\[
i = (1.01317933)^4 - 1 = 5.3769\%
\]

2.3.6 We assume that the NPV of Anne’s cash flows with her father are zero, thus:

\[ 6000v^2 + 8000v^4 - 12000 = 0 \]

This can be solved using either the quadratic equation or the CF function. In either case, we get \( i = 5.067\% \) and \( v = 0.951771 \). We then solve,

\[ 15000v^T = 12000 \]
\[ T = \frac{\ln(4/5)}{\ln v} \]
\[ = 4.514 \]

2.3.10 This can be solved using the IRR and CF functions, and it’s a good opportunity to understand what exactly the calculator is doing. For simplicity, let’s assume all the payment frequencies are equal to 1 (F1 =1, F2=1, etc.). Let \( T \) be the interval between payments and let \( i_T \) be the effective rate for the interval. One can also think of \( i_T \) as the IRR. That is, computing the IRR is the same as solving the equation

\[
\sum_k CF_k (1 + i_T)^{-k} = 0
\]
....for $i_T$. Note that the equation above is equivalent to the equation

$$\sum_k CF_k (1 + i)^{-kT} = \sum_k CF_k v^{kT} = 0$$

where $i$ and $v$ are the usual annual effective and discount rates. Once we have $i_T$ we can use the equation,

$$(1 + i)^T = 1 + i_T$$

to solve for $T$ since we know $i$ and $i_T$.

In this case, we enter,
- CF0 = 300
- C01 = -25
- F1 = 4
- C02 = -40
- F2 = 6
- IRR, CPT

The calc spits out IRR = 2.12626%. This is $i_T$. By the equation above,

$$T = \frac{\ln(1 + i_T)}{\ln(1 + i)}$$

$$= \frac{\ln 1.021262}{\ln 1.0475}$$

$$= 0.45337$$

2.4.4 Cash flows are in 100K units.

<table>
<thead>
<tr>
<th>Firm / t</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>8</td>
<td>-11.5</td>
<td>3.5</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-22</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>32</td>
<td>0</td>
<td>0</td>
<td>-32</td>
</tr>
</tbody>
</table>

A’s and C’s IRR can be calculated algebraically since they only have two cash flows. All four IRR’s can be calculated using the IRR function. For example, for Firm B,
- CF0 = 10
- C01 = 0
- F1 = 1
- C02 = 8
- F2 = 1
2.4.8  

Our interval is a quarter, so when we compute the IRR and get 1.751583, we must keep in mind that this is the quarterly effective rate. The annual effective rate is 

\[(1.01751583)^4 - 1 = 7.1926\%\]

2.5.1 First consider Angela. The 6000 grows to

\[6000(1.06)^1.5 = 6548.02\]

by the end of the three year period. We could therefore view her as having two cash flows: \(-8000\) at \(t = 0\) and \(10548.02 = 6548.02 + 4000\) at \(t = 3\). To find her yield, one just needs to solve

\[8000(1 + i)^3 = 10548.02\]
for \( i \). Just take some logarithms to get \( i = 9.654633\% \). Kathy’s effective rate satisfies the equation

\[
4000v^3 + 6000v^{1.5} - 8000 = 0
\]

This can be solved using the quadratic equation via the substitution \( y = v^{1.5} \). Alternatively, one can use the CF function, remembering to convert from an effective rate over a 1.5 year interval to a one year interval. In either case, we get \( i = 11.3751\% \).

2.5.3 Let \( i_A, i_B \) and \( i_C \) be the yield rates for the three enterprises. Then the equations for each of the yields are,

\[
\begin{align*}
-23000 + (7000(1.05)^2 + 22500)(1 + i_A)^{-4} &= 0 \\
6000 - 7000(1 + i_B)^{-2} &= 0 \\
17000 - 22500(1 + i_C)^{-4} &= 0
\end{align*}
\]

Only the first equation requires some explanation. The $7000 grows for two years at 5%, so it is equivalent to a cash payment of 7000(1.05)^2 received in four years. All three equations are easily solved algebraically.

2.7.1

\[
i_{tw} = \left( \prod (1 + j_k) \right)^{1/4} - 1
\]

\[
= \left( \frac{10600}{9400} \cdot \frac{14400}{13000} \cdot \frac{P}{13400} \right)^{1/4} - 1
\]

\[
= \left( \frac{15264P}{163748000} \right)^{1/4} - 1
\]

2.7.2 a.

\[
i_{tw} = \left( \prod (1 + j_k) \right)^{1/2} - 1
\]

\[
= \left( \frac{2200}{2000} \cdot \frac{2700}{3200} \cdot \frac{2100}{1900} \right)^{1/2} - 1
\]

\[
= 1.2829
\]

b. Here we need to use the CF function to solve the equation:

\[
2000 + 1000(1 + i)^{-1/2} - 800(1 + i)^{-14/12} - 2100(1 + i)^{-2} = 0
\]

So do,

\[
\star \text{ CF0 } = 2000
\]
"C01 = 0
* F01 = 5
* C02 = 1000
* F02 = 1
* C03 = 0
* F03 = 7
* C04 = -800
* F04 = 1
* C05 = 0
* F05 = 9
* C06 = -2100
* F06 = 1

The output is -0.175940. This is the monthly yield. Annualize via the computation,

\[(1 - 0.0017594)^{12} - 1 = -2.09\%\]

c. Arthur managed the fund poorly. He invested before the stock dropped and withdrew before the stock rose. This means the dollar-weighted yield should be lower than the time weighted yield.

2.7.3 Let \(x\) be the unknown ending balance. Then,

\[i_{tw} = 0.16 = \left( \frac{1230}{1205} \cdot \frac{x}{2030} \right) - 1\]

This is an easy algebraic equation. One solves and gets \(x = 2306,938.21\)

2.R.4 Determine the amounts on each loan. Venture Bank must be repaid 1,000,000(1 + 0.06)^12 = 1,195,618.17. The private investor must be repaid 500,000(1 - 0.05)^5 = 646,177.72. Finally, Sports Manufacturing will receive 200,000(1.07)^3 = 245,008.60 from the titanium supplier. The cash flows are summarized in the table.

<table>
<thead>
<tr>
<th>(t)</th>
<th>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1,500,000</td>
</tr>
<tr>
<td>3</td>
<td>-1,195,618.17</td>
</tr>
<tr>
<td>5</td>
<td>-646,177.72</td>
</tr>
<tr>
<td>2</td>
<td>-200,000</td>
</tr>
<tr>
<td>5</td>
<td>245,008.06</td>
</tr>
</tbody>
</table>

Entering these figures into the CF function and computing the IRR gives an IRR of 5.6037%.

2.R.5 a. I like to organize the data in a table.
The time weighted return over the 36 months is,
\[
\frac{28,212 \cdot 15,892 \cdot 30,309}{24,500 \cdot 18,212 \cdot 23,892} - 1 = 27.47\%.
\]

The annualized time weighted return is \((1.2747)^{1/3} - 1 = 8.427\%\).

b. Skip.

c. Enter the following cash flows. Recall that we pretend we withdraw the remaining balance, 30,309, at the end of the period.
\[
\begin{align*}
\ast \text{CF0} &= 24500 \\
\ast \text{C01} &= 0 \\
\ast \text{F1} &= 15 \\
\ast \text{C02} &= -10000 \\
\ast \text{F2} &= 1 \\
\ast \text{C03} &= 0 \\
\ast \text{F3} &= 6 \\
\ast \text{C04} &= 8000 \\
\ast \text{F4} &= 1 \\
\ast \text{C05} &= 0 \\
\ast \text{F5} &= 12 \\
\ast \text{C06} &= -30309 \\
\ast \text{F6} &= 1
\end{align*}
\]

This gives a computed IRR of 0.84957\% and is monthly. The annual dollar weighted return is
\[
(1.0084957)^{12} - 1 = 10.685\%.
\]

d. The dollar weighted return is higher because wise (or lucky) investment choices were made. The investor deposited before the fund went up and withdrew before the fund went down.

3. Chapter 3

3.2.3 So, I’m going to ignore silliness about final payments differing by a few cents from previous payments. First solve for \(A\):
\[
A_{[5]} = 150,000.
\]
It’s straightforward to solve this equation and get $A = 5331.93$. Now, the equation we’re trying to solve is:

$$5331.93s_{10}^{5}(1.045)^8 + Bs_{8}^{4.5} = 150000$$

The first term on the left-hand side is the accumulation of the first ten payments at a 5% rate for ten years, which further accumulates at a rate of 4.5% for eight years. The second term is the accumulation of eight payments at the 4.5% rate. The only variable in the equation is $B$, so, it’s straightforward to solve and get $B = 5823.90$, which is an increase of 491.88 over the 5% contributions.

3.2.4 The most expensive house she can afford is the sum of the down payment and the present value of the non-escrow portion of her monthly payments, thus:

$$13200 + (820 - 240)d_{360}^{5.85/12} = 111,514.93$$

3.2.6 Start by calculating the annuity payment. We have,

$$32000.02 = Pa_{15}^{7\%} = 9.1079P \Rightarrow P = 3513.43$$

Now calculate how much these deposits accumulate to after 15 years:

$$3513.43 \cdot s_{15}^{5\%} = 75814.78$$

That means the first option accumulates to $75814.78 - 1500 = 74314.77$. We have,

$$74314.77 = 32000.02 \left(1 - \frac{d^{(4)}}{4}\right)^{-60}$$

$$0.986055 = 1 - \frac{d^{(4)}}{4}$$

$$d^{(4)} = 5.5779\%$$
3.2.8 Use the information about Sigmund’s loan to calculate the total amount borrowed from Ludwig. Start by calculating the annual effective rate, \( i \).

\[
1 + i = \left( 1 - \frac{d^{(i)}}{4} \right)^{-4} = 1.055873 \Rightarrow i = 5.55873\% 
\]

The amount borrowed is then,

\[
A = 2000a_\dddot{i}^{5.55873\%} = 2000(4.981736) = 9963.47 
\]

The equation for Karl is then,

\[
9963.47 = 3200a_{\dddot{i}} \\
3.113585 = a_{\dddot{i}} 
\]

This cannot be solved algebraically. One way to use the BAII-Plus is, \( N=4, PV = -3.113585, PMT = 1, FV = 0 \). Then do, \( I/Y, CPT \). This says that \( I = 10.832328\% \).

On the other hand,

\[
1 + I = (1 + i)^T \Rightarrow 1.108323 = (1.055873)^T \Rightarrow T = \frac{\ln 1.10832328}{\ln 1.055873} = 1.8917 
\]

3.3.4 The 30 deposits accumulate to

\[
1200\dddot{s}_{\dddot{i}15} = 83712.95 
\]

This is the balance one year after the final deposit. We can view this as the present value of a 20 year annuity due, giving us the equation:

\[
B \cdot \dddot{a}_{\dddot{i}14} = 93172.95. 
\]

Solving this equation for \( B \), which represent the payment amount, gives \( B = 5922.83 \).

3.3.5 First calculate the accumulation of the first 36 payments (yes, it really is 36 and not 35). Recall that \( \dddot{s}_{\dddot{i}1} \) is the accumulation one year after the final payment. So the accumulated amount at the last payment is:

\[
(1.05)^{-1} \cdot 7500\dddot{s}_{\dddot{i}15} = 718,772. 
\]

After an additional five years, this accumulates to \( C = 718772.45(1 + j)^5 \). Now, \( C \) is the present value of an annuity due with 300 payments of $5800
at an effective monthly rate (calculate this) of 0.3273%. Thus,  
\[ C = 5800 \ddot{a}_{300} |_{0.3273} \]  
718772.45(1 + j)^5 = 1,110,816.39

It’s now straightforward algebra to solve for \( j = 9.09\% \).

3.3.6 We’re given,  
\[ 12 = \ddot{a}_m = \frac{1 - v^n}{d} \]  
\[ 21 = \ddot{a}_n = \frac{1 - v^{2n}}{d} \]

Divide the first equation by the second to get  
\[ \frac{7}{4} = \frac{1 - v^{2n}}{1 - v^n} = \frac{(1 - v^n)(1 + v^n)}{1 - v^n} = 1 + v^n \Rightarrow v^n = 3/4 \]

Sub into the first equation and solve for \( d \).  
\[ 12 = \frac{1 - (3/4)}{d} \]  
\[ d = 1/48 \Rightarrow v = 47/48 \Rightarrow i = 1/47 \]

Then  
\[ a_{\ddot{a}_m} = \frac{1 - v^{4n}}{i} = \frac{1 - (3/4)^4}{1/47} = 32.12891 \]

3.4.3 a. First, determine how much an perpetuity-due costs at an effective rate of 7% with payments of $5000. Note that \( i = 7\% \) implies \( d \approx 6.5421\%. \)

\[ 5000 \ddot{a}_{\ddot{a}_m} |_{7\%} = 5000/d = 76428.57 \]

How long will it take for the 40K to accumulate to this amount?

\[ 40000(1.07)^n = 76428.57 \Rightarrow n = \frac{\ln 1.910714}{\ln 1.07} = 9.56976 \]

This means there will be enough midway through 1989, so the first payment can occur on Jan 1, 1990.
b. But we’ll have more than we need in the account by the tenth year. We will have

\[ 40000(1.07)^{10} = 78686.05 \]

in the tenth year and

\[ 40000(1.07)^{9} = 73538.37 \]

in the ninth year. But we only need \( \frac{76428.57}{1.07} = 71428.57 \) in the ninth year. We therefore have an extra amount of \( 73538.37 - 71428.57 = 2109.80 \).

3.5.2 Let’s say the amount received by each of the three research charities for the first \( n \) years is \( P \) so that the total amount disbursed over the first \( n \) years is \( 3P \). After the first \( n \) years, \( 3P \) will be given to the children’s fund. Let \( i = 12.25\% \). The present value of the payments given to a single research charity is the \( Pa_{\overline{n}|i} \) and the present value of the payments given to the children’s fund is \( 3Pa_{\overline{\infty}|i} \). Note that the children’s fund is a deferred perpetuity. Equating the PV’s gives us,

\[
\frac{1 - v^n}{i} = 3\frac{v^n}{i}
\]

\[ 4v^n = 1 \]

\[ (1.1225)^{-n} = 0.25 \]

\[ n = -\frac{\ln 0.25}{\ln 1.1225} \]

\[ n = 12 \]

Thus, \( n \) is 12 years. For the second part of the problem, since we’re only interested in a proportion, we may as well assume \( P = 1 \). If we change \( i \) to 6\%, then the PV of the children’s fund is

\[
3v^n \cdot \frac{1}{i} = \frac{3(1.06)^{-12}}{0.06} = 24.848468
\]

the PV of the payments to all three research funds is

\[ 3a_{\overline{12}|6} = 25.1515 \]

Thus, the proportion going to the children’s fund is,

\[
\frac{24.848468}{24.848468 + 25.1515} = 49.696\%
\]
3.5.3 One can view this as an annuity-due deferred by 12 years or an annuity immediate deferred by 11 years. In the first case, the equation to solve is,

\[ 21092.04 = P v^{12} \bar{a}_{12|i} \]

and in the latter,

\[ 21092.04 = P v^{11} \bar{a}_{12|i} \]

In either case, \( i \) is known to be 7.8\% so the only unknown is \( P \).

3.5.4 Writing the two equations in thousands gives us,

\[ 2a_{\overline{2n}|i} + a_{\overline{m}|i} = 52.8 \]
\[ 4v^n a_{\overline{m}|i} = 27.4 \Rightarrow v^n a_{\overline{m}|i} = 6.85 \]

Let’s do some algebra (difference of squares!) on the first equation.

\[
52.8 = 2 \cdot \frac{1 - v^{2n}}{i} + a_{\overline{m}|i} \\
= 2 \cdot \frac{(1 - v^n)(1 + v^n)}{i} + a_{\overline{m}|i} \\
= 2(1 + v^n)a_{\overline{m}|i} + a_{\overline{m}|i} \\
52.8 = (3 + 2v^n)a_{\overline{m}|i}
\]

Dividing this last equation by \( v^n a_{\overline{m}|i} = 6.85 \) gives us,

\[
7.70803 = \frac{(3 + 2v^n)}{v^n} \\
7.70803v^n - v^n = 3 \\
5.70803v^n = 3 \\
v^n = 0.525575
\]

Now, since 6.85 = \( v^n a_{\overline{m}|i} = 0.525575a_{\overline{m}|i} \), it must be that \( a_{\overline{m}|i} = 13.0333 \) and

\[
6.85 = (0.525575) \cdot \frac{1 - 0.525575}{i} \Rightarrow \\
i = 3.64\% \Rightarrow \\
v = 1/1.0364 \approx 0.96487
\]
Finally, solve for \( n \) using the equation \( v^n = 0.525575 \).

\[
n = \frac{\ln 0.525575}{\ln 0.96487} = 18
\]

3.6.1  
a) The outstanding loan balance is the present value of the future payments.

\[
OLB = 1516 a_{|16\%} = 5253.10
\]

b) Same argument as in part (a). Note that we can view the final four payments as a deferred annuity.

\[
OLB = 822 a_{|6\%} + (1.06)^{-3}(1516)_{|6\%}
\]
\[
= 2197.22 + (1.06)^{-3}(5253.10)
\]
\[
= 6607.82
\]

3.6.3  
We need to calculate the outstanding loan balance immediately after the 100th payment. This is the amount Mr. Bell owes the bank. At this point, 80 more payments are due, all at 6%. An important detail is we need to calculate the equivalent effective monthly rate. This is,

\[
(1.06)^{1/12} - 1 = 0.486755\%
\]

Thus,

\[
OLB = 1692 a_{|80\%0.486755\%} = 111,894.78
\]

Mr Bell receives \( 258,000 - 111,894.78 = 146,105.22 \) Note that we did not use the 4% rate nor the down payment amount; a lot of unnecessary information in this problem.

3.6.5  
The initial loan is \( 18300 - 3800 = 14,500 \). The effective monthly rate is

\[
(1.052)^{1/12} - 1 = 0.42333\%
\]

so the payments are found by solving the equation

\[
14500 = Pa_{|70\%0.42333}
\]

Using the calculator or straightforward algebra gives \( P = 234.0602 \). Since the problem says the last payment is slightly reduced, we round this up to 234.07. To calculate the remaining balance after the 24th payment, one can use the retrospective method.

\[
OLB_{24} = 14500(1.004233)^{24} - 234.07 a_{|70\%0.42333}
\]
\[
= 16047.18 - 5899.75 = 10147.33
\]
3.6.7 Start by calculating her level payments assuming no missed payments. The
effective monthly interest rate is $i^{(12)}/12 = 0.4\%$.

$$1450.97 = P a_{240.4\%} \Rightarrow P = 635.45$$

Immediately after the eighth payment, her loan balance is

$$635.27 a_{230.4\%} = 9826.25$$

so at the time of the ninth payment, her balance is

$$9826.93(1.004) = 9865.56$$

and this must be paid back in 15 installments. The new payment amount
satisfies the equation

$$9865.56 = Q a_{150.4\%}$$

so $Q = 678.95$.

3.7.2 We decompose the annuity-due into its two parts. The value of the $100$
payments is easy

$$100 \overline{a}_{30:6} = 1215.81$$

To deal with the $300$ payments, it's helpful to write out the series of
present values explicitly.

$$300v^{1/2} + 300v^{3/2} + \ldots + 300v^{39/2} = 300v^{1/2}(1 + v + \ldots + v^{19}) = 300v^{1/2} \overline{a}_{20:6}$$

$$= 300(1.06^{-1/2})(12.1581)$$

$$= 3542.70$$

The total value is thus $3542.70 + 1215.81 = 4758.51$.

3.7.5 Throughout this problem, let $i$ and $v$ be the monthly effective interest rate
and discount factor.

$$\frac{1}{12}$$

$$i = (1.05)^{1/2} - 1 = 0.071\%$$

$$v = 1/(1.004074) = 0.99594$$
We can calculate the initial loan amount, $A$, using the information about Sean’s loan.

$$A = 200a_{48|0.4074} + 500v_{48}a_{48|0.4074}$$

$$= (200 + 500v_{48})a_{48|0.4074}$$

$$= (200 + 411.35)(43.518)$$

$$= 26604.79$$

Now enter Karl’s cash flows into the CF function,
- CFo = -26604.79
- C1 = 7000
- F1 = 3
- C2 = 3000
- F2 = 2
- C3 = 0
- F3 = 1
- C4 = 2000
- F4 = 1

Compute the IRR. This gives you 3.1043% and is the effective rate over time period $T$. Now solve for $T$.

$$(1 + i)^T = 1.031043$$

$$T = \frac{\ln 1.031043}{\ln 1.004074}$$

$$= 7.5192$$

Since we used the monthly effective rate, $i$, $T$ is 7.5192 months or 7.5192/12 $\approx$ 0.627 years.

3.7.6 First calculate the effective interest rate.

$$i = d/(1 - d) = 0.1/0.9 = 11.11\%$$

Equating the present values gives,

$$1000a_{12|11.11\%} + 2000(1.1111)^{-12}(\ddot{a}_{12|11.11\%}) = Qa_{12|11.11\%} + 3Q(1.1111)^{-20} \cdot \frac{1}{0.1111}$$

$$7175.70 + 4800.77 = 7.9058Q + 3.2826Q$$

$$11976.47 = 11.1884Q$$

$$Q = 1070.44$$
3.8.1 The PV of Al’s annuity is given by the formula,
\[
PV = 100 \cdot \frac{1 - (1-0.04)^{15}}{0.05 - (-0.04)}
\]
\[
= 821.386
\]
We know that the two annuities have the same PV. Since the accumulated value is \((1 + i)^n \times PV\), we have,
\[
FV = (1.05)^n(821.386) = 1626.29
\]
\[
1.05^n = 1.98
\]
\[
n = \frac{\ln 1.98}{\ln 1.05}
\]
\[
= 14
\]

3.8.2 For problems like this, I like to write out the PV series explicitly and work from there. For the \$3000 payments, the present value is given by
\[
3000 \left((1.05)^{-4}(1.04)^{-4.5} + (1.05)^{-5}(1.04)^{-5.5} + \ldots\right) =
3000(1.05)^{-4}(1.04)^{-4.5} \left(1 + (1.05)^{-1}(1.04)^{-1} + (1.05)^{-2}(1.04)^{-2} + \ldots\right)
\]
Letting \(j = (1.05)^{-1}(1.04)^{-1}\) gives us,
\[
3000(1.05)^{-4}(1.04)^{-4.5} \left(1 + j + j^2 + \ldots\right) =
3000(1.05)^{-4}1.04^{-4.5} \frac{1}{1 - j} = 24555.50
\]
Similarly, one can show that the PV of the \$1200 payments is,
\[
1200(1.04)^{-5}(1.05)^{-4.5} \cdot \frac{1}{1 - j} = 9399.34
\]
The total PV is 9399.34 + 24555.50 = 33954.84.

3.8.3 This is covered in greater detail in chapter 4, nonetheless, we can proceed by writing out the series explicitly. Let \(j\) be the monthly effective rate and \(w\) the monthly discount factor. Let \(i\) and \(v\) be the usual annual effective rate and discount factor. Note that \(w^{12} = v\).
\[
1000(w + w^2 + ... + w^{12}) + 1000(1.02)(w^{13} + ... + w^{24}) + ... + 1000(1.02)^{19}(w^{229} + ... + w^{240})
\]
\[
= 1000\left(a_{\ddot{10}2} + (1.02)w^{12}a_{\ddot{10}2} + ... + (1.02)^{19}w^{228}a_{\ddot{10}2}\right)
\]
\[
= 1000a_{\ddot{10}2}(1 + 1.02v + 1.02^2v^2 + ... + 1.02^{19}v^{19})
\]
\[
= 1000 \cdot a_{\ddot{10}2} \cdot \frac{(1.02v)^{20} - 1}{1.02v - 1}
\]

Now, \(j = 0.327\%\) and \(v = 0.9615\), so the PV is,
\[
1000(11748.50) \cdot \frac{0.3218}{0.0192} = 196,614.90
\]

3.9.2 At the end of the ten years, there will be $50,000 in the primary fund, and some unknown amount in the reinvested account. Using formula (3.9.5), and being careful since \(i\) in the problem is the interest rate in the primary fund, not the secondary fund, we have:
\[
100000 = 50000 + 5000i(1s)_{\ddot{10}10\%} \Rightarrow
\]
\[
\frac{10}{i} = (1s)_{\ddot{10}10\%} \Rightarrow
\]
\[
\frac{8_{\ddot{10}10\%} - 11}{0.1} = 75.31667 \Rightarrow
\]
\[
i = 13.278\%
\]

3.9.3 This payment stream can be decomposed into an arithmetically increasing annuity-due and a deferred perpetuity. Use formula 3.9.12 with \(P = 320\), \(n = 22\), \(Q = 30\) and \(d = i/(i + 1) = 3.846\%\) for the arithmetic component. We have,
\[
\left(320\ddot{a}_{\ddot{10}4} + \frac{30}{0.03846} (a_{\ddot{10}4} - 22(1.04)^{-22})\right) + 1.04^{-21}\frac{980}{0.04} =
\]
\[
4809.33 + 780.03(14.45 - 9.2830) + 10751.91 = 19591.87
\]

3.9.5 First, calculate the accumulated balance at the end of six years in the secondary account, which is structured as an arithmetically increasing series
of payments. So, the accumulation at the end of six years is,

$$11000(0.075)(s_{\overline{6}|5\%}) = 825 \cdot \frac{s_{\overline{6}|5\%} - 7}{0.05}$$

$$= 825(22.84) = 18,843$$

There are two more steps. The amount we just calculated will accumulate over the next seven years, and there will be seven more interest deposits of 66000(0.075) = 4950. Thus, the total accumulation in the secondary account after 13 year is,

$$18843(1.05)^7 + 4950s_{\overline{7}|5\%} = 26513.99 + 40302.94$$

$$= 66,816.93$$

Finally, we add the amount in the primary account giving us

$$66816.93 + 66000 = 132,816.83.$$ 

3.11.1 First, we solve for \( n \) using logarithms (or the TVM functions) using the equation,

$$200000 = 25000e_{\overline{n|1\%}}$$

This gives \( n \approx 8.37978 \) and tell us that \( f = 0.37978 \). Using formula 3.11.3 for the drop payment, we have

$$25000 \left( \frac{(1.01)^{0.37978} - 1}{0.01} \right) (1.01)^{1-0.037978} = 25000(0.37861)(1.00619)$$

$$= 9523.86$$

3.R.6 Let \( i \) and \( v \) be the usual variables. It will be convenient to let \( w = v^2 \) and let \( j = (1+i)^2 - 1 \). Note that \( w \) and \( j \) are just the biannual discount factor and effective rates, respectively. We have,

\[
\begin{align*}
  i &= 0.04 \\
  v &= 0.9615 \\
  j &= 0.0816 \\
  w &= 0.9246
\end{align*}
\]
The PV of the cash flow in thousands is then,

\[
v^7 \left[ (10v + 15v^2 + \ldots + 10v^{29} + 15v^{30}) + 10v^{31} + 10v^{32} + \ldots \right]
\]

\[
= 10v^8 (1 + v^2 + \ldots + v^{28}) + 15v^9 (1 + v^2 + \ldots + v^{28}) + 10v^{37} (v + v^2 + \ldots)
\]

\[
= (10v^8 + 15v^9) (1 + w + \ldots + w^{14}) + \frac{10v^{37}}{i}
\]

\[
= (10v^8 + 15v^9) \left( \frac{1 - w^{15}}{1 - w} \right) + \frac{10v^{37}}{i}
\]

\[
= (7.3069 + 10.5388)(9.16817) + 58.5742
\]

\[
= 163.612 + 58.5742
\]

\[
= 222.186
\]

Since we did everything in thousands, the answer is about $222,186.

3.R.7 Let \( g = 0.03 \) and \( i = 0.042 \). As usual, \( v = 1/1.042 \). Let \( x = (1 + g)v \). By writing out the series and using the usual formula for geometric series, the present value of the first 24 payments is

\[
2000 \left( \frac{x^{24} - 1}{x - 1} \right) = 42148.845
\]

The value of these 24 payments at time 50 is

\[
42148.845(1.042)^{50} = 329740
\]

The 26 level payments accumulate to

\[
40005\frac{5}{14.2} = 189,989.09.
\]

Summing the two gives a total future value of,

\[
189989.09 + 329740 = 519729.09.
\]

4. Chapter 4

4.2.2 First, let’s do some easy interest rate conversions. We’re given that \( d^{(4)} = 3\% \). Using chapter 1 techniques, we get the annual variables: \( d = 2.9664\% \) and \( v = 0.970336 \). It’s also helpful to calculate the monthly effective rate, call this \( j \). It’s not hard to see \( j = 0.2513\% \). Finally, let’s introduce the variable \( u \) to be the monthly discount factor, \( u = 1/(1+j) \). The idea is to equate the values of the two annuities. Let’s call the payment amount of
the annual annuity $P$ and say the annuity is for a $k$ year term. Then,

$$1000a_{135|j} = P\tilde{a}_{j|135}$$

$$1000 \cdot \frac{1 - \nu^{12k}}{j} = P \cdot \frac{1 - j^k}{d} \Rightarrow$$

$$1000 \cdot \frac{1 - j^k}{j} = P \cdot \frac{1 - j^k}{d} \Rightarrow$$

$$P = \frac{1000d}{j} = 11806.30.$$  

4.2.3 First, determine the amount in the savings account at the sale date. Call the biannual rate $I$. Then $I = (1.06)^2 - 1 = 12.36\%$. The accumulation in the savings account is then

$$1000\tilde{s}_{\nu|12.36} = 7189.29$$

The discount rate $D$ corresponding to $I$ is $D = I/(I + 1) = 11\%$. The value of the perpetuity at the sale date is

$$1000/D = 9090.90$$

so she has a total of $7189.29 + 9090.90 = 16280$ to invest. The triannual interest rate is $J = (1.06)^3 - 1 = 19.10\%$ so $P$ is given by the equation

$$16280 = Pa_{30|19.10} \Rightarrow P = 3765$$  

4.2.4 We are given,

$$12692.59 = 1000a_{Y|10}$$

$$1069.12 = 300a_{Y|3}$$

The second equation allows us to solve for $i$. It is, $i = 4.785$. We can now solve for $Y$ in the first equation and get $Y = 20$. The biannual rate $I$ is $(1 + i)^2 - 1 = 9.790\%$. Thus, the answer is

$$500a_{100|9.790} = 3099.$$  

4.2.5 Solve this using the equation on page 182.

$$4769.30 = 1000\tilde{a}_{p|1}$$

$$= 1000\frac{a_{72|3,6575}}{a_{72|3,6575}}$$

The numerator of the right hand side can be calculated. Doing some algebra leaves us with,

$$a_{3|6,575} = \frac{5\times1000}{5,3097}$$
Solve for $k$ using the calculator. Set $I/Y = 3.6575$, $PV = -5.301097$, $PMT = 1.0 = FV$ then do CPT, $N = 6.000$.

4.4.3 I'll do this one using the formula on p.191 with one small adjustment. The variables are $P = 15,000$, $Q = 4000$, $i = 5\%$, $k = 3$, $r = 10$ and $n = 30$. Now, we need to account for the fact that the first payment is 5 years from now using the formula in the book would give us the PV if the first payment were 3 years from now. This means we need to multiply the formula by $(1.05)^{-2}$, so

$$PV = (1.05)^{-2} \left( 15000 \frac{a_{3|35}}{s_{3|35}} + \frac{4000}{(0.05)s_{3|35}} \left( \frac{a_{3|35}}{s_{3|35}} - 10(1.05)^{-30} \right) \right)$$

$$= (1.05)^{-2} \left( \frac{230587}{3.1525} + \frac{4000}{0.157625} \left( \frac{15.37}{3.1525} - 2.313 \right) \right)$$

$$= (0.907029) (73144 + 25377(2.5626))$$

$$= 125,326$$

4.5.2 We use the formula that should be on page 196. Note that $i^{(12)} = 2.9595\%$.

$$PV = m \left[ P_{n/m} + \frac{Q}{i^{(m)}} (a_{m|n} - m) \right]$$

$$= 12 \left[ 120d_{3|93}^{(12)} + \frac{5}{0.029595} (a_{3|93} - 30(1.03)^{30}) \right]$$

$$= 12 [2384.24 + 168.947(19.6 - 12.36)]$$

$$= 43290$$

4.5.4 This one is simple enough to avoid fancy formulas. Just calculate the accumulation of each set of 12 payments separately and sum them. The monthly effective rate is $j = 0.47252\%$.

$$1200(1 + i)^5 + 1300(1 + i)^4 + 1400(1 + i)^3 + 1500(1 + i)^2 + 1600(1 + i) =$$

$$= (1200(1.0582)^5 + 1300(1.0582)^4 + 1400(1.0582)^3 + 1500(1.0582)^2) s_{12|j} + 1600(1.0582)^2 = 82984$$

4.6.2 Use Fact 4.6.9 which states that the present value of a continuously paying annuity from time 0 to time $n$ at a rate of $f(t)$ is

$$\int_0^n f(t)v(t) \, dt$$
In this case, $f(t) = 50$. We need to find $v(t)$. Recall from chapter 1 that $a(t) = e^t \int_0^t \delta(r) \, dr$ and $v(t) = 1/a(t)$. We have,

$$v(t) = \frac{1}{a(t)} = e^{-\int_0^t \delta(r) \, dr} = e^{\int_0^t (1+r)^{-1} \, dr} = e^{-\ln(1+t)} = (1 + t)^{-1}$$

Now use Fact 4.6.9.

$$PV = \int_0^n f(t)v(t) \, dt = 50 \int_0^{10} (1 + t)^{-1} \, dt = 50 \ln 11 = 119.89$$