In this case, $f(t) = 50$. We need to find $v(t)$. Recall from chapter 1 that $a(t) = e^{\int_0^t \delta(r) \, dr}$ and $v(t) = \frac{1}{a(t)}$. We have,

$$v(t) = \frac{1}{a(t)} = e^{-\int_0^t \delta(r) \, dr} = e^{-\int_0^t (1 + r) \, dr} = e^{-\ln(1 + t)} = \frac{1}{1 + t}.$$ 

Now use Fact 4.6.9.

$$PV = \int_0^n f(t) \, v(t) \, dt = 50 \int_0^1 (1 + t) \, dt = 50 \ln(1 + 1) = 119.89.$$ 

5. Chapter 5

5.2.1 The interest in the payment at $t = 3$ is

$$22342 \left[(1 + i)^2 - 1\right]$$

giving us,

$$22345 \left[(1 + i)^2 - 1\right] = 1916.14$$

$$(1 + i)^2 = 1.085764$$

$$i = 4.2\%$$

We can now determine the initial loan balance; call this amount $B$. We have,

$$(1.042)B - 8000 = 22,342 \Rightarrow B = 29119$$

The amount of interest in the first payment is $iB = 1223$ and the principal is $8000 - 1223 = 6777$. The payment at time 3, call it $P_3$, is given by the equation

$$22342(1.042)^2 - P_3 = 9908 \Rightarrow P_3 = 14350.14$$

The principal in this payment is $14350.14 - 1916.14 = 12434$. The interest in the final payment is $0.042(9908) = 416.14$ and the principal is 9908. The amount of the last payment is $(1.042)(9908) = 10324.14$. 


5.2.2 By the amortization schedule on page 218, the principal in the $k$th payment is $Qv^{n-k+1}$. The first step is to solve for the interest rate, $i$. We have the following:

$$259.34 = Qv^{360-82+1} = Qv^{279}$$
$$230.19 = Qv^{360-56+1} = Qv^{305}$$

I’ll leave it to you to solve this system of equations for $v$ (it’s just algebra). Keep in mind that $v$ is a monthly discount factor, so, once you get it you can find the monthly effective rate. This is 0.45965%.

Now, the formula for the interest, also on p. 218, in the $k$th payment is

$$Q(1 - v^{n-k+1})$$

So, you need to compute $Q$, the amount of each payment. You can do this, for example, by solving for $Q$ in the equation

$$259.34 = Qv^{279}$$

You should get $Q = \$932.27$. Now you have all the info you need:

$$\text{int}(133) = Q(1 - v^{360-133+1}) = (932.27)(0.64852) = 604.59.$$ 

5.2.3 This is a good question because it forces you to ask, what concepts in Chapter 5.2 apply to all amortized loans, and which apply only to loans with level payments? Let’s let $PMT(k)$ be the amount of the $k$th payment, $\text{int}(k)$ the interest portion, and $\text{Prin}(k)$ the principal portion. So, for any loan we have:

$$PMT(k) = \text{Int}(k) + \text{Prin}(k).$$

It’s also always true that the interest portion depends on the outstanding loan balance immediately after the previous payment:

$$\text{Int}(k) = i \cdot OLB_{k-1}$$

so I’ll solve this problem using

$$\text{Prin}(14) = PMT(14) - i \cdot OLB_{13}$$

But $PMT(14) = 400 + 13(45) = 985$. $OLB_{13}$ is a little trickier. This is the present value of 12 arithmetically increasing future payments- the first of which is 985. We can can therefore use formula 3.9.4 with $P = 985$, $Q = 45$, $n = 12$, $i = 4\%$ to obtain

$$OLB_{13} = (I_{985,45u})_{T4} = 11370.44$$
Thus our solution to the problem is,

\[ Prin(14) = 985 - (0.04)(11370.44) \]
\[ = 530.18 \]

5.2.4 I’ll start by doing part (c), calculating the balloon payment \( B \). We can break the loan amount into three chunks: the payments at the first rate, the payments at the second rate, and the balloon payment. Note that the monthly effective rates are \( 0.06/12 = 0.005 \) and \( 0.075/12 = 0.00625 \).

\[
117,134.80 = 988.45a_{50|0.5\%} + 988.45(1.005)^{-60} \cdot a_{120|0.625\%} + B(1.005)^{-60}(1.00625)^{-120} \\
= 51128.13 + 61735.34 + 0.351017B \Rightarrow \\
B = 12168.44 \quad [B \text{ is the extra payment in addition to the last 988.45 payment due, so the total final/balloon payment is } 12168.44+988.45=13156.89] \\
\]

The outstanding loan balance after the 84th payment is the present value of future payments. There are \( 180 - 84 = 96 \) payments of 988.45 and the balloon payment still outstanding. Hence,

\[
OLB(84) = 988.45a_{50|0.625\%} + (1.00625)^{-96}(12168.44) \\
= 71194.12 + 6690.66 \\
= 77884.78 \\
\]

Now let’s solve part (b). First, note that by a similar calculation to the one above, we get \( OLB(83) = 78383.34 \) meaning the principal in the 84th payment is

\[ OLB(83) - OLB(84) = 498.56 \]

The interest is therefore,

\[ 988.45 - 498.55 = 489.90. \]

5.2.5 Our approach depends on how seriously we take the fact that the 180th payment is slightly more ($1.72) than the other payments. It turns out if we assume all payments are level, our answer is only a few cents different than if we incorporate this difference, so, let’s just ignore it. It’s straightforward to get the amount of each payment is \( Q = 2434.14 \). Let’s call the monthly interest rate and discount factor \( i \) and \( v \). Then the amount of interest in the first payment is \( Q(1-v^{180}) \), \( Q(1-v^{179}) \) in the second, etc. The amount
of interest in the first 60 payments is therefore,
\[ Q \left( (1 - v^{180}) + (1 - v^{179}) + \ldots + (1 - v^{121}) \right) = Q \left( 60 - v^{121}(1 + v + \ldots + v^{59}) \right) \]
\[ = Q \left( 60 - v^{121} \frac{v^{60} - 1}{v - 1} \right) \]
\[ = 2434.14 \left( 60 - 0.5663734 \cdot \frac{0.24565}{0.004687} \right) \]
\[ = 73797.64 \]

5.3.1 Clearly, the first sinking fund deposit is 5200 implying a net loan balance of 14000 – 5200 = 8800. Also, the interest rate on the loan is 889/14000 = 6.35%. 889 will also be the interest on the loan for year 2. For year 4, the interest is 14000((1.0635)^2 - 1) = 1834.45. The interest rate on the sinking fund is 218.40/5200 = 4.2%. The SF balance after 2 years is 5200 + 218.40 + 3000 = 8418.40. The loan balance after 2 years is 14000 – 8418.40 = 5581.60. The final sinking fund balance should be 14000. Call the last SF deposit \( P \). Then,
\[ 8418.40(1.042)^2 + P = 14000 \]
\[ P = 4859.60 \]

The SF interest is 8418.40(1.042^2 - 1) = 722.

5.3.2 Note that there’s a lot of useless information in this problem– this is common of actuarial problems. The equation to solve here is,
\[ D \cdot s_{\text{3j}} \cdot (1 + j)^{10} + 2D \cdot s_{\text{3j}} = 238000 \]
where \( j \) is the effective semiannual rate of the sinking fund. The first term is the accumulation of the first six payments which accumulates like an annuity for three years, and then compounds for an additional five years. The second term is the accumulation of the contributions made in the final five years. Calculating \( j \) is easy:
\[ j = \left( 1 + \frac{0.042}{12} \right)^6 - 1 \approx 2.1185\% \]
Substituting in to the equation gives us,
\[ D \left( (6.327)(1.233) + 2(11) \right) = 238000 \Rightarrow \]
\[ D \approx 7980 \]

5.3.3 During the first six years, the interest portion of Alan’s payment is 18,000 \times (0.08/4) = 360 meaning the contribution to the sinking fund is 770 – 360 = 410. Similarly, during the final two years, the interest portion is
18000 \times (0.12/4) = 540 and the sinking fund contribution is 230. The balance in the sinking fund at the end of eight years is,

\begin{align*}
410s_{\overline{15}\rvert} \cdot (1.015)^8 + 230s_{\overline{15}\rvert} &= 410(28.633)(1.1265) + 230(8.4328) \\
&= 15164.
\end{align*}

That means the sinking fund is short by 18000 − 15164 = 2836.

5.3.4 Good algebra problem. The equations are,

\begin{align*}
(P - 13500i)s_{\overline{4}\rvert} &= 13500 \\
(1.2P - 13500(2i))s_{\overline{4}\rvert} &= 13500
\end{align*}

Since \( s_{\overline{4}\rvert} = 15.026 \), we have...

\begin{align*}
P - 13500i &= 898.45 \\
1.2P - 27000i &= 898.45
\end{align*}

Solving this easy linear system gives \( P = 1123 \).

5.4.1 First we need to calculate the quarterly effective rate, call this \( i \). Using chapter 1 techniques, we get \( i = 1.0101\% \). We can use the formula on page 143 to determine the amount of the first payment, \( P \):

\begin{align*}
39,999.85 &= P \left( \frac{1 - \left( \frac{1.02}{1.0101} \right)^{32}}{0.0101 - 0.02} \right) \\
&= 37P \Rightarrow \\
P &= 1081.08
\end{align*}

Now, the balance immediately after the 20th payment is the PV of the 12 future payments. We can use the same formula again, but this time the "initial payment" is the 21st payment: 1081.08(1.02)^{20} = 1606.43. So the outstanding loan balance is,

\begin{align*}
OLB_{20} &= (1606.43) \left( \frac{1 - \left( \frac{1.02}{1.0101} \right)^{12}}{0.0101 - 0.02} \right) = \$20,147
\end{align*}

Now, we can also calculate the balance after the 19th payment.

\begin{align*}
OLB_{19} &= (1081.08)(1.02)^{19} \left( \frac{1 - \left( \frac{1.02}{1.0101} \right)^{13}}{0.0101 - 0.02} \right) = (1574.93)(13.655) = 21,505.69
\end{align*}

This means the interest in the 20th payment is 21505.69 \cdot i = 217.20. The total principal paid is therefore 1574.93 − 217.20 = 1357.73

5.4.2 It’s helpful in problems like this to write out the first couple years to see what the pattern will be. Let’s set \( B = 12500 \). Then the interest earned
in the first period is $Bi$ and the first payment is $2Bi$. This means that the balance immediately after the first payment is,

$$OLB_1 = (1 + i)B - 2iB = (1 - i)B.$$  

A similar analysis shows that $OLB_2 = (1 - i)^2 \cdot B$ and $OLB_k = (1 - i)^k \cdot B$. Now we’re interested in knowing when the balance immediately prior to a payment is less than 1800. The bal. immediately before the $k$th payment is $(1 + i) \cdot OLB_{k-1}$. We have the equation,

$$B(1 + i)(1 - i)^n = 1800$$

$$12500(1.085)(0.915)^n = 1800$$

$$n \approx 22.73$$

We should round this up to 23. Now, by the same formula, the amount due immediately before the 24th payment is,

$$12500(1.085)(0.915)^{23} = 1758.02.$$  

5.4.5 Let $P$ be the amount of the initial payment. It’s relatively straightforward to see that the present value of the payments, which must total 100,000, is given by the following.

$$100,000 = P \left( \frac{1}{1.02} + \frac{1.1}{1.02^2} + \frac{1.1^2}{(1.02^2)(1.06)} + \ldots + \frac{1.1^{11}}{(1.02)^2(1.06)^{10}} \right)$$  

$$= P \left( \frac{1}{1.02} + \frac{1.1}{1.02^2} + (1.02)^{-2}(1.06) \left( \frac{1.1^2}{1.06^2} + \ldots + \frac{1.1^{11}}{1.06^{11}} \right) \right)$$  

$$= P \left( \frac{1}{1.02} + \frac{1.1}{1.02^2} + (1.02)^{-2}(1.06) \frac{1.1^2}{1.06^2} \left( \frac{1 - (1.1/1.06)^{10}}{1 - (1.1/1.06)} \right) \right)$$  

$$= P \left( 0.9803 + 1.0573 (0.972) \frac{0.4483}{0.037736} \right)$$  

$$= P \approx 15.0856 \cdot P$$

$$P = 6634.34$$

This means that immediately after his 5th payment, his OLB is,

$$P(1.1)^5(1.06)^{-1} + P(1.1)^6(1.05)^{-2} + \ldots + P(1.1)^{11}(1.06)^{-7} = P(1.1)^4 \left( \frac{1.1}{1.06} + \ldots + \frac{1.1^7}{1.06^7} \right)$$  

$$= P(1.1)^4 \cdot \frac{1.1}{1.06} \left( \frac{1 - (1.1/1.06)^7}{1 - (1.1/1.06)} \right)$$  

$$= (6634.34)(1.1)^4 \frac{1.1}{1.06} \frac{0.296}{0.037736}$$  

$$= 79068.68$$
5.R.1 Each monthly total payment consists of three sub-payments: the level payment for the 6% component of the loan, the interest payment for the 5% component, and the sinking fund deposit. The payment for the 6% part is easy to find by solving the equation

\[ Ba_{\text{F6\%}} = 50000 \]

This implies \( B = 5963.79 \). Now, for the other half of the money, Dustin pays 2500 = 0.05 \times 50000 in interest, so all that remains is the calculation of the sinking fund deposit. Call the initial deposit \( P \). The deposits must accumulate to 50,000 so we have,

\[
50000 = P(1.03)^{11} + P(1.04)(1.03)^{10} + \ldots + P(1.04)^{11}
\]

\[
= P \left( 1.03^{11} + (1.04)(1.03)^2 + \ldots + 1.04^{11} \right)
\]

To get a geometric series, it’s helpful to divide both sides of the equation by 1.03\(^{11}\), giving...

\[
50000(1.03)^{-11} = P \left( 1 + \frac{1.04}{1.03} + \ldots + \frac{1.04^{11}}{1.03^{11}} \right)
\]

\[
= P \left( 1 - \frac{(1.04/1.03)^{12}}{1 - (1.04/1.03)} \right) \Rightarrow
\]

\[
36121.05 = 12.6618P \Rightarrow
\]

\[
P = 2852.76
\]

That means the amount of the fifth sinking fund deposit is

\[
2852.76(1.04)^4 = 3337.32
\]

The sum of the three components is,

\[
3337.32 + 2500 + 5963.79 = 11801.
\]

5.R.2 By the chapter, the interest and principal components are, \( I_k = Q(1 - v^{n-k+1}) \) and \( P_k = Qv^{n-k+1} \). Here, \( n = 80 \), and \( v \) is the monthly effective discount factor. Exercise: show \( v = 0.98554 \). So, we just have to solve:

\[
I_k = P_k
\]

\[
Q(1 - v^{81-k}) = Qv^{81-k}
\]

\[
2v^{81-k} = 1
\]

\[
81 - k = \frac{\ln(1/2)}{\ln v}
\]

\[
k = 33.4
\]
Rounding to the nearest quarter gives \( k = 33 \).

5.R.9 Let \( j \) be the monthly effective rate and \( i \) the annual effective rate. Then, \( j = 0.375\% \) and \( i = 4.594\% \). Working with the formula on p.196 gives us,

\[
B_{39} = 130a_{39j} + 140(1 + j)^{-9}a_{39j} + 150(1 + j)^{-21}a_{39j}
\]
\[
= 1149.33 + 11.7125((140)(0.96687) + (150)(0.92441))
\]
\[
= 1149.33 + 11.7125(135.362 + 138.662)
\]
\[
= 4358.84
\]

Now, the amount of the 40th payment is 130, so by the formula on page 226, we have:

\[
B_{40} - jB_{39} = 130 - jB_{39}
\]
\[
= 130 - (0.00375)(4358.84)
\]
\[
= 113.65
\]

5.R.10 His initial monthly payments are given by,

\[
P a_{360|0.5625\%} = 126523 \Rightarrow P = 820.63
\]

His outstanding loan balance immediately after the 60th payment is,

\[
820.63a_{360|0.5625\%} = 118,774.36
\]

If he increases his payments to 1120.63, let \( m \) be the number of months it will take to pay off his loan. We must solve,

\[
1120.63a_{m|0.5625\%} = 118,774.36
\]

The solution is \( m = 161.66 \) months. It’s unfortunate that this isn’t closer to an integer, but, we our formulas still work. Namely, the interest in the \( k \)th payment is still \( Q(1 - v^{n-k+1}) \) when \( n \) is not an integer.

So we may as well assume we have a 161.66 month loan where each monthly payment is 1120.63. Call this \( Q \). Interest in this second year is,

\[
Q(1 - v^{161.66-13+1}) + \ldots + Q(1 - v^{161.66-24+1}) = Q \left( 12 - (v^{149.66} + \ldots + v^{138.66}) \right)
\]
\[
= Q(12 - v^{137.66}(v + v^2 + \ldots + v^{12}))
\]
\[
= 1120.63(12 - 0.462a_{12|0.5625\%})
\]
\[
= 1120.63(12 - 0.462(11.5725))
\]
\[
= 7456.10
\]
6. Chapter 6

6.2.1 So, we can’t really avoid calculator use on this one. Below are two solutions, one of which is the same as the other one on the webpage. It’s always good to know what equation we are solving, it is:

\[ 50a_{\overline{20}|i} + 1020v^{20} = 880 \]

The problem is you can’t isolate \( v \) or \( i \).
- \( N = 20 \) (20 coupon payments)
- \( PV = -880 \)
- \( PMT = 50 \)
- \( FV = 1020 \)
- \( CPT \ I/Y \)

Here’s one way to do it with the bond worksheet, although it’s a little awkward since I’m entering it as a 20 year bond with annual payments and coupons of 5%.
- \( SDT = 12-31-1990 \) (default)
- \( CPN = 5 \)
- \( RDT = 12-31-2010 \)
- \( RV = 102 \)
- 360 (default)
- \( 1/Y \)
- \( PRI = 88 \)
- \( CPT \ YLD \)

6.2.2 Solvable with or without the calculator. The formula is,

\[ 2590 = 162.50a_{\overline{16}|4} + C(1.054)^{-8} \]

162.5 is the coupon payment, calculated as follows: \( 2500 \times 0.065 = 162.5 \).

6.2.3 Note that the effective semiannual yield is 2.5%. Let \( k \) be the number of coupons of the second bond. Equate the prices (present values) of the two bonds:

\[ PV_1 = PV_2 \]
\[ 30a_{\overline{16}|2.5} + 1000(1.025)^{-16} = 27.5a_{\overline{20}|2.5} + 1000(1.025)^{-2k} \]
\[ 1065.27 = 27.5a_{\overline{20}|2.5} + 1000(1.025)^{-2k} \]

Now use the TVM worksheet to solve for \( 2k \).
- \( I/Y = 2.5 \)
- \( PV = 1065.27 \)
- \( PMT = 27.5 \)
– FV = 1000
– CPT N
You get \( N = 2k = 42.835 \), so to the nearest half-year, \( k = 21.5 \).

6.2.4 Let \( i \) be the effective semiannual yield, \( v = 1/(1 + i) \) and let \( n = 2k \), so that \( k \) is the number of coupon payments. The bond equations are:

\[
2318.63 = 100a_{21\%} + 2000v^{2k}
\]
\[
2531.05 = 110a_{21\%} + 2000v^{2k}
\]

Subtracting one equation from the other gives us \( a_{21\%} = 21.242 \). Subbing this into the first equation gives us,

\[
2318.63 = 2124.20 + 2000v^{2k} \Rightarrow v^{2k} = 0.097215
\]

Writing out the annuity symbol gives,

\[
2318.63 = 100 \cdot \frac{1 - v^{2k}}{i} + 2000v^{2k}
\]

Subbing in \( v^{2k} \) leaves us with just one variable, \( i \). This works out to \( i = 4.2369\% \). We should multiply this by two to get the nominal rate of 8.5%. Now solve for \( k \):

\[
0.097215 = v^{2k} = (1.042369)^{-2k} \Rightarrow 2k \approx 56 \Rightarrow n \approx 28.
\]

6.2.5 Let’s identify the variables of the first bond:

\[
F = C = 3000
\]
\[
r = g = 6\%
\]
\[
Fr = Cg = 180
\]
\[
n = 28
\]
\[
j = 3\%
\]

The proceeds of the sale of this bond is just another way of asking for the price, which is:

\[
P = 180a_{25\%} + 3000(1.03)^{-28} = 4688.77
\]
or you could use the calc to get this. The equation for the second bond is
\[
4688.77 = 0.04Fa_{\overline{20}|3} + F(1.03)^{-20} \\
= (0.5951)F + (0.55368)F \\
= 1.1488F \Rightarrow \\
F = 4081.54
\]

6.2.6 Recall that the price of a bond is the present value of all payments– the coupon and the final redemption amount. So for this problem, we’ve got three sets of payments: five years of $35 coupons, five years of unknown coupons, and a redemption payment of $1100. Now, the semiannual effective yield is
\[
j = 1.0735^{1/2} - 1 = 3.61\
\]
Thus,
\[
982 = 35a_{\overline{10}|3.61} + \frac{1000q}{2}a_{\overline{10}|3.61}(1.0735)^{-5} + 1100(1.0735)^{-10} \\
= 289.47 + 2900q + 541.22 \Rightarrow \\
151.31 = 2900q \Rightarrow \\
q = 5.2\
\]

6.3.1 The discount formula is,
\[
C - P = C(j - g)a_{\overline{n}|i}
\]
Let’s sub in what we know:
\[
57 = C(0.091 - g)a_{\overline{10}|9.1}
\]
To deal with \(g\), use the fact that \(Cg = Fr = 270\), hence \(g = 270/C\). So,
\[
57 = C \left(0.091 - \frac{270}{C}\right) (7.125) \\
8 = 0.091C - 270 \\
C = 3054.95
\]
Since \(C - P = 57\), \(P = 2997.95\).
6.3.2 It’s good practice to "write out the alphabet" of variables given in the problem.

\[ F = 2000 \]
\[ r = 5.5\% \]
\[ Fr = Cg = 110 \]
\[ n = 20 \]
\[ j = 5.2\%/2 = 2.6\% \]

By the premium/discount formula,

\[ -83.28 = C \left( \frac{110}{C} - 0.026 \right) a_{\overline{20}|2.6} \]
\[ -83.28 = (110 - 0.026C)(15.443) \Rightarrow C = 4438.18 \]

Note that the problem could go further and ask you to calculate \( g \) and \( P \). Exercise: do this.

6.3.3 Another problem with lots of extraneous information. The premium is,
\[ P - C = 1400 - 1100 = 300 \]. Thus,

\[ Ds_{\overline{10}|8} = 300 \]
\[ 14.487D = 300 \]
\[ D = 20.71 \]

6.5.1 The price can be computed by hand OR using the TVM worksheet OR using the BOND worksheet. So, you’ve got options.

\[ P = 65a_{\overline{14}|4} + 2100(1.04)^{-30} \]
\[ = 1123.98 + 647.47 \]
\[ = 1771.45 \]

The amount for the adjustment of principal is:

\[ P_{10} = B_9 - B_{10} \]
\[ = C(g - j)v^{n-10+1} \]
\[ = 2100 \left( \frac{65}{2100} - 0.04 \right) (1.04)^{-21} \]
\[ = -8.34 \] Note: The amount for accumulation of discount = 8.34 (the negative of the adjustment of principal amount in the case of a discount bond---see p.256 of text)
It’s important to keep track of the sign; when the bond is priced at a discount, \( P_t \) is always negative. For the final part of the problem,

\[
I_t + P_t = Cg \Rightarrow \\
I_{10} = 65 - (-8.34) \\
= 73.34
\]

Note that if you mess up the sign in the previous part of the problem, you’d get this part wrong.

6.5.2 Calculate the price of the bond (using any of the usual methods).

\[
P = 360d_{3\%} + 6000(1.03)^{-20} \\
= 8677.86
\]

This means the premium is \( 8677.86 - 6000 = 2677.85 \). The amortization of premium is,

\[
P_7 = C(g - j)v^{n-7+1} \\
= 6000(0.03)(1.03)^{-14} \\
= 119
\]

6.5.3 We are given the following.

\[
P_2 = 977.19 = C(g - j)v^{29} \\
P_4 = 1046.9 = C(g - j)v^{27}
\]

Dividing one equation by the other gives us,

\[
1.071225 = v^{-2} \Rightarrow \\
v = 0.966184 \\
j = 3.5\%
\]

Subbing this back into the equation for \( P_2 \) gives us,

\[
977.19 = C(g - j)(0.36875) \Rightarrow \\
C(g - j) = 2650.02
\]

Now, we can solve for the premium:

\[
P - C = C(g - j)a_{3\%3.5} \\
= (2650.02)(18.392) \\
= 48739.16
\]
6.9.1 a. We need to solve for the yield, $y_k$, that corresponds to a call at time $t = k$. The bond formula that we need to solve is therefore:

$$51248 = 4000a_{\overline{k}|k} + (50000 + 300(10 - k))(1 + y_k)^k$$

For example, when there’s a call at time $t = 7$, use the TVM function to solve the equation

$$51248 = 4000a_{\overline{7}|_7} + 50900(1 + y_7)^{-7}$$

It turns out $y_7 = 7.7284\%$. Solve for the other yields similarly to get,

- $y_6 = 7.7923\%$
- $y_7 = 7.7284\%$
- $y_8 = 7.6849\%$
- $y_9 = 7.6549\%$
- $y_{10} = 7.6341\%$

Clearly, the minimal yield is $y_6$ and the maximal yield is $y_{10}$.

b. One way to do this is to see how much money will accumulate after ten years in the account at $r = 6\%$ in each of the five scenarios. These accumulations are calculated below.

$$
\begin{align*}
(4000s_{\overline{6}|6} + 51200)(1.06)^4 &= 99863 \\
(4000s_{\overline{7}|6} + 50900)(1.06)^3 &= 100611 \\
(4000s_{\overline{8}|6} + 50600)(1.06)^2 &= 101337 \\
(4000s_{\overline{9}|6} + 50300)(1.06) &= 102041 \\
(4000s_{\overline{10}|6} + 50000) &= 102723
\end{align*}
$$

The highest yield, call it $y_M$, corresponds to the highest accumulation. So, we solve:

$$51248(1 + y_M)^{10} = 102723$$

$$10 \ln(1 + y_M) = \ln \frac{102723}{51248}$$

$$y_M = 7.2011\%$$

Similarly, solve for the minimal yield $y_m$.

$$51248(1 + y_m)^{10} = 99863$$

$$y_m = 6.8988\%$$
6.9.3  

a. Start by determining the price of the bond. This is given by,

\[ P = 420a_{\overline{8}|6} + 6000(1.066)^{-8} = 6145.56 \]

Note that the bond is purchased at a premium. You might initially think that since the bond is purchased at a premium, the lowest yield of calls at years 5, 6, 7 would occur at year 5. This reasoning is incorrect (think carefully about why). In fact, the minimum yield of years 5, 6, 7 would occur at year 7. That means that we only need calculate her yield if called at the end of two or seven years. For a call at two years, we have:

\[ 6145.56 = 420a_{\overline{2}|6} + 6300(1 + y_2)^{-2} \Rightarrow y_2 = 8.042\% \]

For a call at five years, we have

\[ 6145.56 = 420a_{\overline{7}|6} + 6200(1 + y_7)^{-7} \Rightarrow y_7 = 6.937\% \]

The minimum of the yields 8.042%, 6.937% and 6.6% is 6.6%.

b. Suppose the bond is called after at \( t = 2 \). There have been two coupon payments of 420. A payment of 6300 is invested for six years at a rate of 5.5% which is equivalent to a single payment of 6300(1.055)^6 = 8686.71 at \( t = 8 \). We can therefore use the cash flow function with the following values:

* CF0 = -6145.56  
* CF1 = 420  
* F1 = 2  
* CF2 = 0  
* F2 = 5  
* CF3 = 8686.71  
* F3 = 1

One solves and gets an IRR = 6.17807. We analyze the other cases similarly (it turns out the one we just did has the minimal yield).

6.R.1  We have the formula \( I_t = jB_{t-1} \). In this case, \( 52.89 = I_1 = 0.05B_0 = 0.05P \Rightarrow P = 1057.80 \)

This seems to imply the premium is 57.80. To get the more precise answer of 57.86, note that if we use the data we have so far and solve for \( n \), we get \( n = 6.99 \). If we assume \( n \) is an integer and round it to 7, then we use the data \( N = 7 \), \( I/Y = 5 \), \( PMT = 60 \), \( FV = 1000 \) and solve to get \( PV = 1057.86 \). This is a bit circular.

7. Chapter 7  

There’s nothing really new here- this is just a perpetuity. The effective quarterly rate is \( j = 1.062^{1/4} - 1 = 1.515\% \). The stock price is just the
value of a perpetuity paying $0.28 per quarter, thus:

\[
\text{Price} = \frac{0.28}{j} = 18.48
\]

8. CHAPER 8

8.3.1 First calculate the spot rates.

\[
r_1 = \frac{10000}{9765} - 1 = 2.41\%
\]

\[
r_2 = \left(\frac{10000}{9428}\right)^{1/2} - 1 = 2.99\%
\]

\[
r_3 = \left(\frac{10000}{8986.82}\right)^{1/3} - 1 = 3.63\%
\]

Now calculate the forward rates. There are several ways to do this. Some small discrepancies due to rounding.

\[
f_{[0,1]} = r_1 = 2.41\%
\]

\[
f_{[0,2]} = r_2 = 2.99\%
\]

\[
f_{[0,3]} = r_3 = 3.63\%
\]

\[
1 + f_{[1,2]} = \frac{(1 + r_2)^2}{1 + r_1} = \frac{(1.0299)^2}{1.0241} \Rightarrow f_{[1,2]} = 3.57\%
\]

\[
(1 + f_{[1,3]})^2 = \frac{(1 + r_3)^3}{1 + r_1} = \frac{(1.0363)^3}{1.0241} \Rightarrow f_{[1,3]} = 4.24\%
\]

\[
1 + f_{[2,3]} = \frac{(1 + r_3)^3}{(1 + r_2)^2} = \frac{(1.0363)^3}{(1.0299)^2} \Rightarrow f_{[2,3]} = 4.91\%
\]

8.3.2 The zero coupon bond tells us \( r_1 \).

\[
r_1 = \frac{1000}{974} - 1 = 2.669\%
\]

Now use the two year bond to calculate \( r_2 \) and the three year to calculate \( r_3 \).

\[
988 = 30(1.02669)^{-1} + 1030(1 + r_2)^{-2} \Rightarrow
\]

\[
(1 + r_2)^{-2} = 0.931 \Rightarrow
\]

\[
r_2 = 3.648\%
\]
990 = 40(1.02669)^{-1} + 40(1.03648)^{-2} + 1040(1 + r_3)^{-3}
990 = 38.96 + 37.234 + 1040(1 + r_3)^{-3}
0.87866 = (1 + r_3)^{-3} \Rightarrow \ r_3 = 4.4062\%

8.3.3 The yields of the discount bonds are the spot rates, hence:

\[
\text{Price} = 30(1.019)^{-1/2} + 30(1.023)^{-1} + 30(1.0265)^{-3/2} + 1030(1.0305)^{-2}
\]
\[
= 29.719 + 29.326 + 28.846 + 969.932
\]
\[
= 1057.82
\]

8.3.4 Let’s start by calculating the price of the coupon bond. We may as well assume the bond has a face value of $1000 so that the coupons are each $100 (you can assume the face value is anything you want, or leave it as a variable if you wish). It’s priced to yield 3% so

\[
P = 100(1.03)^{-1} + 1100(1.03)^{-2} = 1133.95
\]

If all the rates are synchronized correctly, then the price of this bond should be the same when priced using the spot rates. If not, there’s an opportunity for profit. We’re given \( r_1 = 3.2\% \) and \( r_1 = 1.8\% \). So the price of the coupon bond using these rates is

\[
P = 100(1.018)^{-1} + 1100(1.032)^{-2} = 1131.07
\]

Since the market price is higher than the prices implied by the spot rates, the market price is too high. This tells us what our strategy should be: we should always sell high, so the first thing we should do is sell the coupon bond. More specifically:

1. Sell one unit of the two year coupon bond. This gives you a cash balance of 1133.95. In two years, you must repay $1100 and in one year you must pay $100.
2. To finance the $100 payment in one year, invest \( \frac{100}{(1 + r_1)} = 98.23 \)
3. Invest the remaining \( $1035.72 \) in a 2 year zero coupon bond. This is the same as putting the money in a saving account for 2 years at a rate of \( r_2 = 3.2\% \).
4. Your cash balance is now zero.
5. After 1 year, the \( $98.23 \) has grown to $100, so you pay the coupon of the two-year coupon bond. Your cash balance is still zero.
6. At \( t = 2 \), the money in the 2-year bank account (or zero coupon bond) has grown to \( (1035.72)(1.032)^2 = 1163.07 \). You pay the coupon and principal of the 2-year coupon bond, a total of $1100. This leaves you with a guaranteed profit of $63.07.
8.3.5 This is a typical bootstrapping problem. We need to calculate \( r_3 \) and \( r_5 \).

The one year bond gives us \( r_1 = y_1 = 1.435\% \).

\[
40(1.01435)^{-1} + 1040(1 + r_2)^{-2} = 40(1.02842)^{-1} + 1040(1.02842)^{-2}
\]
\[
39.434 + 1040(1 + r_2)^{-2} = 38.8946 + 983.314 \Rightarrow r_2 = 2.8703\%
\]

\[
40(1.01435)^{-1} + 40(1.028703)^{-2} + 1040(1 + r_3)^{-3} =
\]
\[
40(1.03624)^{-1} + 40(1.03624)^{-2} + 1040(1.03624)^{-3} =
\]
\[
39.434 + 37.799 + 1040(1 + r_3)^{-3} = 1010.509 \Rightarrow r_3 = 3.6751\%
\]

\[
40(1.01435)^{-1} + 40(1.028703)^{-2} + 40(1.036751)^{-3} + 1040(1 + r_4)^{-4} =
\]
\[
40(1.03943)^{-1} + 40(1.03943)^{-2} + 40(1.03943)^{-3} + 1040(1.03943)^{-4} =
\]
\[
39.434 + 37.799 + 35.8952 + 1040(1 + r_4)^{-4} = 1002.208 \Rightarrow r_4 = 3.9976\%
\]

\[
40(1.01435)^{-1} + 40(1.028703)^{-2} + 40(1.036751)^{-3} + 40(1.039976)^{-4} + 1040(1 + r_5)^{-5} =
\]
\[
40(1.04683)^{-1} + 40(1.04683)^{-2} + 40(1.04683)^{-3} + 40(1.04683)^{-4} + 1040(1.04683)^{-5} \Rightarrow
\]
\[
39.434 + 37.799 + 35.8952 + 34.1953 + 1040(1 + r_5)^{-5} = 970.169 \Rightarrow r_5 = 4.7956\%
\]

\[
(1 + f_{[3,5]})^2 = \frac{(1.047956)^5}{(1.036751)^3} = 1.134205 \Rightarrow f_{[3,5]} = 6.499\%
\]

8.3.6

Price = \( 180(1.045)^{-1} + 180(1.055)^{-2} + 180(1.055)^{-3} + 3180(1.06)^{-4} \)
\[
= 172.249 + 161.721 + 153.29 + 2518.858
\]
\[
= 3006.12
\]

May as well use the calc to get the yield. Use,

- \( N = 4 \)
- \( PV = -3006.12 \)
- \( PMT = 180 \)
- \( FV = 3000 \)

CPT I/Y gives you 5.9412\%. 
9. Chapter 9

9.1.1 First calculate the price of the coupon bond.

\[
\text{Price} = 30(1.02)^{-1} + 1030(1.02)^{-2} = 29.412 + 990 = 1019.42
\]

The price of the zero coupon bond is \(1000(1.030225)^{-1/2} = 985.22\). Let’s say you purchase \(a\) dollars of the zero coupon bond and \(b\) dollars of the coupon bond. This leads to the following system of equations:

\[
\begin{align*}
10000 &= b \cdot \frac{1030}{1019.42} \\
5000 &= a \cdot \frac{1000}{985.22} + b \cdot \frac{30}{1019.42}
\end{align*}
\]

This system has the solution \(a = 4639.14\) and \(b = 9897.28\). The total initial investment is \(a + b = 14,536.42\).

9.1.2 Suppose that the end of two years \(i = 18\%\). Then the proceeds from the two year bond are reinvested and accumulate to

\[
(24881.52)(1.18) = 29360.19.
\]

Now, the six year bond will pay \(7451.62(1.055)^6 = 10274.61\). At the end of two years, you sell the six year bond, which can now be viewed as a four year bond priced to yield 18% implying a price of

\[
10274.61/(1.18)^4 = 5299.53
\]

The total at the end of three years is thus \(29360.19 + (5299.53)(1.18)^3 = 35,613.64\). Now let’s say \(i = 1\%\). Tracing through the calculations we just did, the total proceeds are

\[
(24881.52)(1.01) + 10,274.61/(1.01)^4 = 25130.33 + (9873.70)(1.01) = 35,102.77
\]

9.2.1 First calculate the price of the bond; you’ve done this a lot by now, so, I’ll leave out the calculation.

\[
P(i) = P(0.08) = 958.43
\]

Now calculate the duration.
\[ P(0.08) \cdot D(0.08, \infty) = 1200(1.08)^{-10} \cdot 10 + \sum_{j=1}^{10} 60(1.08)^{-j} \cdot j \]
\[ = 5558.32 + 60(Ia)_{108} \]
\[ = 5558.32 + 60 \cdot \frac{\partial P}{\partial i} \bigg|_{i=0.08} - 10(1.08)^{-10} \]
\[ = 5558.32 + 60(32.688) = 7519.60 \Rightarrow 
D(0.08, \infty) = \frac{7519.60}{958.43} = 7.8457 \]

9.2.2
\[ D(0.07, 1) = -\frac{1}{P(0.07)} \cdot \frac{\partial P}{\partial i} \bigg|_{i=0.07} \]
\[ = 7443.81/1312 = 5.6736 \]
\[ D(0.07, \infty) = D(0.07, 1) \cdot 1.07 \]
\[ = 6.0708 \]

9.2.3 The modified duration of an \( N \) year zero coupon bond is \( N/(1 + i) \), thus
\[ D(0.05, 1) = 8/1.05 = 7.69105. \]

9.2.4 We have the Taylor approximation,
\[ P(i) = P(i_0) + P'(i_0)(i - i_0) = P(i_0) - P(i_0)D(i, 1)(i - i_0) \]
\[ P(0.044) = 1120.58 - 1120.58(0.0015)D(0.0425, 1) \]

We need to calculate \( D(0.0425, 1) \).
\[ D(i, 1) = P(i_0) \left[ 1 + \frac{i_0(2)}{2} \right] / (1 + i) \]
\[ = 3.58(1.02103)/1.0425 \]
\[ = 3.50627 \]

So,
\[ P(0.044) = 1120.58 - 1120.58(0.0015)(3.50627) = 1114.69 \]

A similar calculation solves the second part of the problem.
9.2.5 The effective semiannual rate is $j = 1.05^{1/2} - 1 = 2.4695\%$. The cash flow is the same as a perpetuity so the price is $P = 50/0.024695 = 2024.70$.

\[
P \cdot D(0.05, \infty) = 50 \left( \frac{1}{2} (1 + j)^{-1} + (1 + j)^{-2} + \frac{3}{2} (1 + j)^{-3} + \ldots \right) \\
= \frac{50}{2} (Ia)_{\infty j} \\
= 25 \left( \frac{1}{j} + \frac{1}{j^2} \right) \\
= 25 (40.494 + 1639.77) = 42006.50 \Rightarrow \]
\[
D(0.05, \infty) = 42006.50/2024.70 = 20.747
\]

Now, since

\[
D(i, \infty) = D(i, m) \left( 1 + \frac{i^{(m)}}{m} \right)
\]

we have,

\[
D(0.05, 2) = D(0.05, \infty) \cdot (1.024695)^{-1} = 20.247
\]

9.2.6 This is mathematically the same as a perpetuity with geometrically increasing payments. Using techniques of the previous chapter, it’s easy to show that

\[
P(i) = 100/(i - 0.02) \\
P(0.06) = 100/0.04 = 2500 \\
P'(i) = -100(i - 0.02)^2 \\
P'(0.06) = -62500 \\
D(0.06, 1) = -P'(0.06)/P(0.06) = 62500/2500 = 25 \\
D(0.06, \infty) = (1.06)(25) = 26.5
\]
9.2.9 Start by calculating the durations of the individual bonds. The effective monthly rate is \( r = 0.721\% \). For the five year annuity, we have:

\[
D^5(0.09, \infty) = \frac{1000 \sum_{k=1}^{60} \frac{k}{12} (1.09)^{-k/12}}{1000 \sum_{k=1}^{60} 1.09^{-k/12}} = \frac{(1/12) (1a)_{0.09/12}}{\bar{a}_{0.09/12}}.
\]

\[
= \frac{(48.92 - 60(1 + r)^{-60})/(0.00721)(12)}{48.568} = \frac{(48.92 - 39)/0.08652}{48.568} = 2.3627.
\]

The price of this annuity is \( 1000a_{0.09/12} = 48567.58 \). Note that the effective semi-annual rate is \( j = 4.403\% \). The duration of the eight year bond is,

\[
D^8(0.09, \infty) = \frac{8(20000)(1 + j)^{-16} + 800 \sum_{k=1}^{16} (k/2)(1 + j)^{-k}}{20000(1 + j)^{-16} + 800 \sum_{k=1}^{16} (1 + j)^{-k}} = \frac{80298.60 + 400(1a)_{0.09/12}}{19088} = \frac{80298.60 + 400(11.812 - 16(1.04403)^{-16})/0.04403}{19088} = 6.00.
\]

The prices of the two bonds are,

\[
1000a_{5.0} = 3889.65
\]

The duration of the portfolio is a weighted average of the individual durations, hence

\[
D_{\text{portfolio}} = \frac{48567.58}{67656}(2.361) + \frac{19088}{67656}(6.00) = 1.695 + 1.693 = 3.39.
\]

9.2.10 The annual rate is,

\[
i = \left(1 + \frac{0.068}{12}\right)^{12} - 1 = 7.016\%.
\]
and the monthly effective rate is \( j = 0.068/12 = 0.5667\% \). Let \( v = 1/(1+j) \)
Suppose the monthly payment is \( A \). We hope \( A \) will go away because we
don’t know what it is. Then the price (or PV or value) of the mortgage is,

\[
P(i) = A \cdot \frac{a_{360j}}{\frac{1}{v}}
\]

So the duration in months is

\[
D(i, \infty) = \frac{1}{A \cdot \frac{a_{360j}}{\frac{1}{v}}} (A(1+j)^{-1} + 2A(1+j)^{-2} + \ldots + 360A(1+j)^{-360})
\]

\[
= (Ia)_{360j} \cdot \frac{1}{i} \cdot \frac{v}{1-v^{360}}
\]

\[
= \frac{154.255 - 47.075}{0.8692} = 123.309
\]

The duration in years is \( 123.309/12 = 10.276 \).

9.3.1 Straightforward but tedious. Just calculate the weighted average.

9.3.2 See solution to 9.3.1.

9.4.1 Let \( a \) be the amount spent on the 1-year bonds, and \( b \) the amount spent
on the 4-years. The surplus is zero if

\[
a + b = 300000(1.04)^{-3} = 266,698.91
\]

Now, recalling that the Macaulay duration of an \( N \) year zero coupon bond
is \( N \), and the formula for the duration of a portfolio of bonds, we get:

\[
\frac{a}{a+b} + 4 \cdot \frac{b}{a+b} = 3
\]

\[
a + 4b = 3(266,698.91) = 800,096.7
\]

Solving this system gives \( b = 177,799, a = 88,899.64 \). Now, is the convexity
condition satisfied? The convexity of the portfolio of assets is:

\[
\frac{88899.64}{266698.91} \cdot (1)^2 + \frac{177799}{266698.91} \cdot (4)^2 = 11
\]

The convexity of the 3-year liability is \( 3^2 = 9 \). Since \( 11 > 9 \), the convexity
condition is satisfied.
9.4.2 First consider the portfolio of liabilities. The price (value) of the liabilities is:

$$200000(1.04)^{-2} + 500000(1.04)^{-5} = 184911 + 410963 = 595875$$

The duration of the portfolio of liabilities is,

$$\frac{184911}{595875} \cdot (2) + \frac{410963}{595875} \cdot (5) = 4.069$$

The convexity of the liabilities is,

$$\frac{184911}{595875} \cdot (4) + \frac{410963}{595875} \cdot (25) = 18.48$$

Now consider the assets. Let $a$ and $b$ be the amount spent on the two and five-year bonds, respectively. The price, duration and convexity of a $1$ two year bond are,

$$P(0.04) = 0.1(1.04)^{-1} + 1.1(1.04)^{-2}$$

$$= 0.0962 + 1.017 = 1.1132$$

$$D(0.04, \infty) = \frac{1}{1.1132} (0.1(1.04)^{-1} + (2)(1.1)(1.04)^{-2})$$

$$= \frac{1}{1.1132} (0.0962 + 2.034) = 1.9136$$

$$C(0.04, \infty) = \frac{1}{1.1132} (0.1(1.04)^{-1} + (4)(1.1)(1.04)^{-2}) = 3.741$$

The Macaulay duration of the zero coupon bond is 5. Equating the prices and durations of the assets and liabilities gives us a system of equations:

$$a + b = 595875$$

$$\frac{a}{a + b} \cdot (1.9136) + \frac{b}{a + b} \cdot (5) = 4.069$$

Just do some algebra to get $a = 179,743$, $b = 416,138$. To see if the portfolio is immunized, calculate the convexity of the portfolio of assets and compare it to the convexity of the liabilities, 18.48. Since the Macaulay convexity of the five year bond is 25, the Macaulay convexity of the assets is:

$$\frac{179743}{595875} \cdot (3.741) + \frac{416138}{595875} \cdot (25) = 18.5875$$

This is greater than 18.48, so the portfolio is immunized.
To think about: I solved this problem by saying "Set the duration of the assets equal to the duration of the liabilities" and "Compare the convexity of the assets to the convexity of the liabilities." Now, the book suggests asking the similar questions: "Set the duration of the surplus to zero" and "See if the convexity of the surplus is positive." One has to be careful because in general the two sets of questions are not equivalent. They are not equivalent because duration and convexity are nonlinear. They would be equivalent if, for example,

\[ D(\text{bond}1 + \text{bond}2) = D(\text{bond}1) + D(\text{bond}2). \]

Nonetheless, I claim my logic is valid– what about the particular setup of Redington Immunization allows me to answer the rephrased questions?

9.R.1 Let’s say the face value of the bond is $1000 (of course it doesn’t matter what the face is). The price of the bond is then given by:

\[ P = 75a_{15} + 1000(1.09)^{-15} \]
\[ = 604.55 + 274.54 \]
\[ = 878.09 \]

Then we can view the bond as a portfolio consisting of a 15 year annuity immediate and a single payment of $1000 one year from now. We showed in class that the for an annuity-immediate, \( D(i, 1) = v(Ia_m/a_m) \). So the duration of the 15 year annuity component is,

\[ D^{\text{ann}}(9\%, 1) = (1.09)^{-1} \cdot \frac{\ddot{a}_{15} - 15(1.09)^{-15}}{0.09} \cdot \frac{1}{a_{15}} \]
\[ = (1.09)^{-1} \cdot \frac{8.7862 - 4.1181}{0.09} \cdot \frac{1}{8.0607} \]
\[ = 0.9174 \cdot 51.8678 \cdot 0.12406 \]
\[ = 5.903 \]

The $1000 payment can be viewed as a zero coupon bond payment and therefore has \( D^{1000} = 15/(1.09) = 13.761 \). Now, the duration of the portfolio is the weighted duration of the prices, hence

\[ D^{\text{port}} = \frac{604.55}{878.09} \cdot 5.903 + \frac{274.54}{878.09} \cdot 13.761 \]
\[ = 8.357 \]
Note that, as Kong mentions in his solutions, the answer in the back of the book is incorrect (there’s an extra ”8” in there). The same computation work in computing $D(7.5\%, 1)$. 