Introduction. This manual is being written to help actuarial students become more efficient problem solvers for the Part II examination of the Casualty Actuarial Society and the Society of Actuaries—and better investment analysts in the process. The enhanced capabilities of the TI BA II Plus™ calculator allow the automated solution of problems which once required a long series of calculations using specialized formulas and table lookups. Times have changed. In 1962 a friend (a banker, not an actuary) spent 48 hours finding an internal rate of return that the TI BA II Plus™ will now find in a few seconds.¹ Today that same banker expects such problems to be solved immediately using modern technology.

The reader who is studying for actuarial examinations should be aware that the calculator is an accessory and does not replace knowledge of the basic theoretical structure. The exams still require formulas to be memorized and manipulated. The modern technology makes some practical problems much easier to solve. It does not guarantee success on actuarial examinations. It can be helpful.

This is not a calculator key instruction manual. We will review some keystroke basics in the next section, but the ultimate goal is not instruction in how to punch keys. The goal is to illustrate how to be quick when that is possible.

We will assume that the reader has studied (or is studying) the theory of interest and has been introduced to terms such as annuity, net present value, yield and internal rate of return.

¹ The BA 35 calculator has many useful features, but the BA II Plus has a number of useful additions and is substantially more powerful. The internal rate of return feature is one of the most useful of the additions.
1) **Review of features of the calculator.** The TI BA II Plus™ comes with an instructional manual, and we will assume that actuarial students can and will read this. We will review some features of the calculator that are extremely useful.

**Formatting.**

To see format options use the keystrokes

\[
\text{2nd FORMAT} \quad \text{(FORMAT is displayed above the . key)}
\]

You are then able to make format selections. The first format selection shown will be the number of decimal places to display. This is a personal choice, but it is worth noting that entering 9 as your choice will give up to 9 places while showing only the relevant number of decimal places. To try this, use the keystrokes

\[
9 \quad \text{ENTER}
\]

Then $6 \times 9$ will show as 54, while $1/3$ will display as 0.333333333.\(^2\)

You can review other format options by scrolling down using the ↓ key.

The next format selection is DEG for degrees. If you wish to set the calculator to use radians, use the keystrokes

\[
\text{2nd SET} \quad \text{(SET is displayed above the ENTER key)}.
\]

The display will show RAD for radians. You can return to degrees by repeating the keystrokes above. To finalize the selection you desire, scroll down using the ↓ key. The upper right corner of the display will show the indicator RAD if that choice is made. There will be no indicator if DEG is selected.

The next format selection is the United States date format. This can be replaced by a European date format. If you scroll down once more, you can choose between US and European format for numbers. There is no apparent reason for an actuarial student in the United States to use the European format for either dates or numbers.

The last format selection provides the option of using the standard hierarchy of algebraic operations. The default selection is the less desirable Chn selection. This is the chain method in which operations are executed in the order in which

---

\(^2\) If you wish to try this out, hit the CE/C key to get out of the format menu and try those calculations. You can always return.
they are entered. Using 2nd SET, you can select AOS—the standard algebraic operating system. A simple example will illustrate the difference.

**Keystrokes.** \(5+3\times4=\)

**Chn answer.** 32, since the input was read as \((5+3)\times4\)

**AOS answer.** 17, since the input was read as \(5+(3\times4)\) according to the standard hierarchy.

It is appropriate to note here that the TI BA II Plus™ also has parenthesis keys which allow for more appropriate grouping of calculations.

To leave the format menu, press the CE/C key.
Multiple memories

Many calculations rely on combining a number of previously computed numbers. The TI BA II Plus™ has ten memory registers, imaginatively named 0, 1, 2, …, 9. If you wish to save a calculated value, you can store it in any one of these registers instead of writing it down and re-entering it later. For example, if you have calculated the value 3.72018, you can save it in memory register 1 by using the keystrokes

3.72018 STO 1

Once the value is stored, it may be recalled by the keystrokes

RCL 1

Another nice feature of the calculator is that the memories may be reviewed in a simple manner. The keystrokes

2nd MEM (MEM is displayed above the 0 key)

will open a display in which the user can scroll the contents of all memory registers. To leave the memory display, press the CE/C key.
**Time value of money (TVM).**

This capability is also found on the BA 35 calculator, but it has been improved on the TI BA II Plus ™ to provide a more rational sign convention. (This will be discussed further below). The feature provides level payment annuity calculations, using the familiar variables represented by the keys below.

<table>
<thead>
<tr>
<th>Key</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>The number of periods</td>
</tr>
<tr>
<td>I/Y</td>
<td>The interest rate per period (multiplied by 100 to simplify entry)</td>
</tr>
<tr>
<td>PV</td>
<td>The present value of all future payments</td>
</tr>
<tr>
<td>PMT</td>
<td>The level periodic payment</td>
</tr>
<tr>
<td>FV</td>
<td>The future value of all prior cash flows</td>
</tr>
</tbody>
</table>

We will illustrate the use of these keys with a simple example, but it is important to review some key calculator settings first.

**Are payments made at the beginning or end of periods?**

To choose the appropriate setting, use the keystrokes

\[2\text{nd} \ BGN\] (BGN is displayed above the \[\text{PMT}\] key)

The default setting is END for end of period payments. To select BGN for beginning of period payments, use the keystrokes \[2\text{nd} \ SET\] as before. To go back to END mode, use the same keystrokes. To leave the format menu, press the \[\text{CE/C}\] key.

The upper right corner of the display will show the indicator BGN if that mode is selected. No indicator is shown when the END mode is elected.
Setting the BA II Plus for a single payment and conversion per period

There is an option to set the calculator for specialized problems in which the payment period and interest conversion period differ -such as the situation where payments are made quarterly but interest is converted monthly. This feature can be helpful, but our actuarial students have found that it can also be easily confused. We will show how to use it later. To simplify our beginning examples, we will set the calculator so that each period has a single payment and a single interest rate conversion. This can be done using the keystrokes

2nd P/Y (P/Y is displayed above the I/Y key)

You will be prompted for the number of payments per period by the display P/Y =. Set this to 1 with the keystrokes

1 ENTER.

Once this is done, scroll down and you will be prompted for the number of conversions per period by the display C/Y =. Set this to 1 in the same fashion, and leave the format menu by pressing the CE/C key. 3

The calculator should now be in END mode and set to values of 1 for P/Y and C/Y. We recommend starting any new problem by clearing the TVM variables in the calculator using the keystrokes

2nd CLR TVM (CLR TVM is displayed above the FV key)

If the variables are not cleared, a calculation may be based in part on values that are left over from the last problem.

3 The calculator default values for P/Y and C/Y are 12. This is useful for mortgage calculations, but confusing to actuarial students who expect the calculator to have a single payment and conversion per period.
Annuity Features of the BA II Plus (TVM)

We will now illustrate the annuity capabilities of the TI BA II Plus™ with a few simple examples.

Example 1. You obtain a loan of $10,000 which will be repaid with equal annual payments for 5 years. The annual interest rate is 8%. Find a) the annual payment and b) the balance at the end of three years.

Solution.

a) Enter the information given and find the payment using the keystrokes

\[
\begin{align*}
5 & \quad \text{N} \\
8 & \quad \text{I/Y} \\
10000 & \quad \text{PV} \\
\text{CPT} & \quad \text{PMT}
\end{align*}
\]

The required payment is displayed as a negative number

\[
\text{PMT} = -2,504.56
\]

b) The balance at the end of 3 years is the balance immediately after the third payment is made. This is equal to the future value of all prior payments paid or received by time 3, and can be simply calculated by the following keystrokes

Using the TI BA II Plus™

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The relevant values of I/Y, PV and PMT do not require entry, since they are already in the appropriate memory registers and have not been cleared. The answer is displayed as

\[ FV = -4,466.30 \]

Note that this too is negative. The FV is a balance which you owe on the loan and must pay out if you wish to terminate the loan. The balance you owe is the future value of your loan obligation (positive) netted against the future value of payments made to date (negative). Note that the balance could also be found by taking the present value of the future payments which have yet to be paid.

**Example 2.** A loan of 10,000 is to be repaid by 6 annual payments of 2,200 made at the end of each year. Find the effective annual interest rate.

**Solution.** This is a problem in which the sign convention must be used carefully. The loan is cash to the borrower, and will be entered as a positive number. The payment is cash from the borrower, and will be entered as a negative number. To begin, clear TVM. Then the keystrokes are:

\[ \begin{align*} 
6 & \quad \text{N} \\
-2200 & \quad \text{PMT} \\
10000 & \quad \text{PV} \\
& \quad \text{CPT I/Y} 
\end{align*} \]

The displayed answer is \( I/Y = 8.55947004 \).

**It is extremely important to follow the sign convention in TVM problems. If both PMT and PV had been entered as positive here, an ERROR message would have resulted from the interest rate calculation.**

---

\(^4\) The TI BA 35 is based on a different TVM model and does not use this consistent sign convention. This is covered in more detail in Section 6.
Amortization of an annuity.

In the previous section we found the balance of a loan after 3 periods. This is part of an amortization schedule, which shows the balance of a level payment loan (annuity) at each point in time along with the amounts of principal and interest paid on the loan in each period. The amortization table for the loan in the last section is given in the next table. We will show how to use the calculator to obtain this table easily, but a few comments are in order first.

<table>
<thead>
<tr>
<th>Time</th>
<th>Payment</th>
<th>Interest Paid</th>
<th>Principal Paid</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000.00</td>
<td></td>
<td></td>
<td>10,000.00</td>
</tr>
<tr>
<td>1</td>
<td>-2,504.56</td>
<td>-800.00</td>
<td>-1,704.56</td>
<td>8,295.44</td>
</tr>
<tr>
<td>2</td>
<td>-2,504.56</td>
<td>-663.63</td>
<td>-1,804.93</td>
<td>6,454.51</td>
</tr>
<tr>
<td>3</td>
<td>-2,504.56</td>
<td>-516.36</td>
<td>-1,988.20</td>
<td>4,466.30</td>
</tr>
<tr>
<td>4</td>
<td>-2,504.56</td>
<td>-357.30</td>
<td>-2,147.26</td>
<td>2,319.04</td>
</tr>
<tr>
<td>5</td>
<td>-2,504.56</td>
<td>-185.52</td>
<td>-2,319.04</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The signs used above are those produced by the TI BA II Plus™. The interest and principal paid are negative, as they should be. The balance is shown as a positive, although it is clearly a negative obligation.

The underlying logic is simple. At the time of the first payment you owe $800 interest on $10,000 at 8%. When you make a payment of 2,504.56, the lender recognizes 800 of it as payment of interest owed, and applies the remaining 1,704.56 to principal. Although the arithmetic is simple, it is helpful to use the calculator to speed up the process.

To use the amortization feature of the TI BA II Plus™, you must first use TVM to set up the annuity as we have just done. The amortization routine uses the values that are in the TVM registers. To start the amortization use the keystrokes

2nd AMORT (AMORT is displayed above the PV key)

The amortization routine allows you to find principal and interest paid over the time span starting at a first period P1 and ending at a second period P2. If you wish to focus on just one period, set both P1 and P2 equal to the value of that period. Thus to see what happens in period 1, set P1 = 1 = P2. The first display asks you to enter the value of P1. Use the keystrokes 1 ENTER and then scroll down and repeat this for P2. Scroll down three more times and you will see the displays

BAL = 8,295.44
PRN = -1,704.56
INT = -800.00

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If you scroll down again, you can set P1=2=P2 and see the results for period 2. It is not necessary to go through the periods in order. You could have looked at period 4 first.

The entry method seems a bit cumbersome, but it provides great flexibility. If you enter P1=1 and P2=3, the display will show the total principal and interest paid over the first 3 periods.

There is a subtle issue here, and it will lead to some penny rounding problems. We have shown the value of the periodic payment rounded to dollars and cents. The calculator actually displayed (and kept in memory) the value

$$PMT = -2504.564546.$$  

It is quicker to solve problems by leaving the full precision value in memory, but a real lender would actually bill for 2,504.56. If this is done, the loan will not be paid off at the planned maturity of 5 years. There will be an unpaid balance of a few cents left after the 5th payment. This will not change the solution of student problems significantly, but can affect a real 30 year mortgage loan. For example, a loan of 100,000 at 1% per month for 360 months has a rounded payment of 1028.61. If you make that rounded payment for 360 months, you will still owe approximately 9.08 after the “last” payment.
Cash flow (CF) analysis with NPV and IRR.

Annuity calculations are only powerful enough to analyze a series of level payments. Most investment opportunities in life are not so simple. Cash flow analysis is needed to handle varying series of payments. Suppose you are offered an investment opportunity in which you must invest $1,000 now and another $100 in one year. In return, you are offered three payments of $120 at the ends of the following 3 years and a final payment of $1,320 one year later. Your investment project looks like this:

<table>
<thead>
<tr>
<th>Paid</th>
<th>Time</th>
<th>Received</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1,000</td>
<td>← 0</td>
<td>CF0</td>
</tr>
<tr>
<td>-100</td>
<td>← 1</td>
<td>C01</td>
</tr>
<tr>
<td>2</td>
<td>→ 120</td>
<td>C02</td>
</tr>
<tr>
<td>3</td>
<td>→ 120</td>
<td>C03</td>
</tr>
<tr>
<td>4</td>
<td>→ 120</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>→ 1,320</td>
<td>C03</td>
</tr>
</tbody>
</table>

This is not a level payment annuity, but it is easy to analyze. Hit the [CF] key to activate the cashflow menu. You will see a prompt for the value of CF0, the initial investment. Key in the value -1000 and press [ENTER]. The natural next step is to scroll down. If you do that, you will be prompted for the value of C01, the payment at time 1. Enter the value -100.6

Scroll down again, and there will be a new prompt – “F01=”. This is a request for the number of times (frequency) that this value is repeated. The default value is 1, and if you scroll past the value of 1 will be assumed with no entry. Scroll down again, and you will be prompted for the value of C02. Enter 1207, but remember that this value is repeated 3 times. Scroll down, and when prompted for F02, enter the value 3. Scroll down again, and enter the value 1320 for C03. The entry is done. We repeat the table and the entry strategy next.

<table>
<thead>
<tr>
<th>Paid</th>
<th>Time</th>
<th>Received</th>
<th>Cashflow</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1,000</td>
<td>← 0</td>
<td>CF0</td>
<td>NA</td>
<td></td>
</tr>
<tr>
<td>-100</td>
<td>← 1</td>
<td>C01</td>
<td>F01=1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>→ 120</td>
<td>C02</td>
<td>F02=3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>→ 120</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>→ 120</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>→ 1,320</td>
<td>C03</td>
<td>F03=1</td>
<td></td>
</tr>
</tbody>
</table>

Remember that it is always a good idea to start by clearing the workspace to remove debris from prior problems.

You must press [ENTER] for the entry to become effective.

Remember, you must press [ENTER] for the entry to become effective.

Using the TI BA II Plus™

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The official strategy endorsed in finance texts is to evaluate this project by finding its net present value (NPV) at your own desired earning rate. Luckily the calculator is designed to implement this directly. Suppose you have a benchmark which requires you to earn at a 12% rate on this project. The NPV at 12% is found using the keystrokes and displays

\[
\begin{align*}
\text{NPV} & \quad 12 \quad \text{ENTER} \\
\downarrow & \\
\text{NPV} & \quad \text{CPT}
\end{align*}
\]

The display will show the answer

\[
\text{NPV} = -82.94
\]

Since the NPV is negative, the project should be rejected.

Business decision makers often use the yield or internal rate of return on the project as the criterion for acceptance or rejection. This can be calculated easily by the keystrokes

\[
\text{IRR} \quad \text{CPT}
\]

The display will show the answer \( \text{IRR} = 10 \).

The yield on the project is 10%. An investor who wishes to earn 12% would be wise to reject it. The benchmark had been missed.

Leave the CF option by hitting the \( \text{CE/C} \) key.
Depreciation

The TI BA II Plus™ has automated depreciation procedures for the straight line, declining balance and sum-of-years-digits methods. A simple example will illustrate the keystrokes.

A machine cost $10,000 and will have a salvage value of $1,000 at the end of its 4 year useful life. The machine will be depreciated using the sum-of-years-digits method. A depreciation schedule for the machine can be generated using the following series of keystrokes:

2nd DEPR (DEPR is displayed above the 4 key)

You will see a display indicating the depreciation method. To choose the sum-of-years-digits method, key in

2nd SET

until you see the choice SYD.

Scroll down and you will be prompted for lifetime (LIF =). Key in

4 ENTER

Scroll down again and leave the entry MO = 1 unchanged. (Choosing a month other than 1 enables you to start depreciation some fraction of the way through a year, and will not be used for problems here).

Scroll down and you will be prompted for cost (CST=). Key in

10000 ENTER

Scroll down and you will be prompted for salvage value (SAL=). Key in

1000 ENTER

Scroll down and you will be prompted for a year (YR=). Key in

1 ENTER

8 Actuarial exams often have questions on the sinking fund method. This method is not widely used in practice, and thus was not implemented on the TI BA II Plus™

Using the TI BA II Plus™

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All the relevant variables have now been entered. Scroll down and you will be shown in succession the values for year 1 of the depreciation amount, remaining book value and remaining depreciable value.\(^9\)

\[
\begin{align*}
\text{DEP} &= 3,600 \\
\text{RBV} &= 6,400 \\
\text{RDV} &= 5,400
\end{align*}
\]

If you scroll down again, you will be returned to the YR = display. You can now enter the value 2 and obtain the second year of the depreciation schedule. The full set of results is shown in the following table.

<table>
<thead>
<tr>
<th>YR</th>
<th>DEP</th>
<th>RBV</th>
<th>RDV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>10,000</td>
<td>9,000</td>
</tr>
<tr>
<td>1</td>
<td>3,600</td>
<td>6,400</td>
<td>5,400</td>
</tr>
<tr>
<td>2</td>
<td>2,700</td>
<td>3,700</td>
<td>2,700</td>
</tr>
<tr>
<td>3</td>
<td>1,800</td>
<td>1,900</td>
<td>900</td>
</tr>
<tr>
<td>4</td>
<td>900</td>
<td>1,000</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^9\) Remember that of the $10,000 value only $9,000 was depreciable.
Using the P/Y option to analyze differing payment and conversion frequencies

We have previously advised setting both payment frequency (P/Y) and conversion frequency (C/Y) to 1. Problems in which payment and conversion frequency differ are easily handled with this option. This is shown in the following example.

**Example.** An annuity pays $1000 at the end of each month for eight years. Find the present value if the rate of interest is a nominal 9% convertible semi-annually.

**Solution without P/Y option.** The payment is monthly, and is made for $N=96$ months. The semi-annual rate of 4.5% is converted to a monthly rate using the conversion

\[ i = (1.045)^{1/6} - 1 \approx 0.007363123 \]

We can find the present value using the keystrokes

\[
\begin{align*}
96 & \quad N \\
0.7363123 & \quad I/Y \\
1000 & \quad PMT \\
\text{CPT} & \quad PV
\end{align*}
\]

The answer is $-68,657.10.$

**TI BA II Plus™** has a feature which enables the user to set payment and conversion periods for a problem like this. In the preceding example, there are 12 payments per year but only two semiannual conversion periods. We could use the calculator options to get the solution in the following way.

**Alternate solution using P/Y option.** We reset the calculator for this by using the keystrokes

\[
\begin{align*}
\text{2nd} & \quad \text{P/Y (P/Y is displayed above the I/Y key)} \\
12 & \quad \text{ENTER}
\end{align*}
\]

Once this is done, scroll down and you will be prompted for the number of conversions per period by the display $C/Y = .$ Set this to 2 in the same fashion, and leave the format menu by pressing the $\text{CE/C}$ key.

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Now the calculator is set to solve the problem directly using the keystrokes:

96 \[ N \]
9 \[ I/Y \]
1000 \[ PMT \]
\[ CPT \] \[ PV \]

The answer is -68,657.10 as before, and a simple calculation step has been avoided.

A potential problem with using this feature is that a hurried user may forget to reset the option for the next problem. The authors have found that this is a major problem for our students, and could be a problem on actuarial examinations where time is of the essence. We keep our own calculators set on \( P/Y = C/Y = 1 \). It would be wise to reset the calculator to those settings before trouble blows in.
Clearing memory

It is important to assure that there are no numbers left in memory from the last problem when you begin a completely new problem. When you begin a new TVM problem, it is wise to begin by keying in

\[ \text{2nd CLR TVM (CLR TVM is displayed above the } \text{FV} \text{ key)} \]

When you begin a new CF or DEPR problem, begin by keying in

\[ \text{2nd CLR WORK (CLR WORK is displayed above the } \text{CE/C} \text{ key)} \]

Useful features not covered here.

Some of the potentially useful features of the TI BA II Plus™ are not covered in this brief document because they are less likely to be needed for actuarial examinations. However some individuals may wish to use them when studying interest theory. These features include:

a) Days between dates. The DATE feature will find the number of days between two dates using actual or 30-360 calculations. The date entry is specialized – e.g., the date 1/01/2049 is entered as the number 1.0149. The range of dates that can be entered appears to go from 1950 to 2049.

b) Bond calculations. The specialized BOND feature allows settlement dates to be entered and things like accrued interest to be found. This material is not currently in the Part 2 exam syllabus. The exam problems on bonds can be solved using the TVM keys.

c) Interest conversions. The ICONV feature will convert a nominal rate to be converted to an effective rate based on the number of conversion periods per year. This calculation is so basic and easy to do that the feature does not save time for an actuarial student.
3) _Use of actuarial notation_.

**Level payment annuities**

The actuarial notation for the present value of an annuity immediate of n periods with a payment of 1 made at the end of each period is \( a_\bar{n} \). The future value is denoted by \( s_\bar{n} \). If the interest rate is \( i \), the values are given by the useful formulas

\[
a_\bar{n} = \frac{1 - v^n}{i}
\]

\[
s_\bar{n} = \frac{(1 + i)^n - 1}{i}
\]

It is a simple task to calculate these values on the TI BA II Plus™ using TVM. For example, suppose that we wish to compute \( a_\bar{n} \) with an interest rate of \( i = 5\% \). Clear TVM and enter the values \( N = 9 \), \( PMT = 1 \) and \( I/Y = 5 \). Then compute PV. The keystrokes are

\[
\begin{align*}
9 & \quad \text{N} \\
5 & \quad \text{I/Y} \\
1 & \quad \text{PMT} \\
\text{CPT} & \quad \text{PV}
\end{align*}
\]

The value is displayed on the calculator as \( PV = -7.107821676 \).

To compute \( s_\bar{n} \) with the same interest rate simply reset PV to 0 and compute FV. The additional keystrokes are

\[
\begin{align*}
0 & \quad \text{PV} \\
\text{CPT} & \quad \text{FV}
\end{align*}
\]

The value is displayed on the calculator as \( FV = -11.02656432 \).

\(^{10}\) Recall that \( v = 1/(1+i) \)

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The values of the annuities due (with payments at the beginning of the period) are denoted by $\dd{a}_n$ and $\dd{s}_n$ and can be calculated in the same fashion with the mode set to BGN.

Although this notation can be quite useful, a person can save time in simple calculations by not using it. For example, suppose you wish to find the amount of the annual payment on a loan of 50,000 for 9 years at an annual interest rate of 5%. Before modern technology changed things, you would use a table in which you could look up $9a$ and then find the payment by calculating

$$\text{PMT} = \frac{\text{PV}}{a_{9i}} = \frac{50,000}{7.107821676} = 7,034.50.$$ 

With modern technology this can be done directly using TVM with PV = 50,000, I/Y = 5 and N=9. The answer is the same, and the intermediate step of writing down the equation is avoided. We mention this because we have seen our students spend extra time on problems by writing down the old table calculation relationships when they are not needed. In this manual we will give many examples of problems where the actuarial notation is superfluous and time can be saved by omitting it. There are also problems in which the notation is essential, and we will indicate some of those too.

**Increasing and decreasing annuities**

The notation for increasing and decreasing annuities is of some importance for actuarial examinations. The present value of the annuity with payments 1,2,...,n is denoted by $(Ia)_n$. The present value of the annuity with payments n, n-1,...,2,1 is denoted by $(Da)_n$. It is traditional to memorize a separate formula for each, but both are special cases of the formula for the present value of a series of n cash flows which start at value P and then change by an amount Q in each subsequent period. The cash flows are:

$$P, P+Q, P+2Q, \ldots, P+(n-1)Q$$

The formula for the present value of this series of cash flows is

$$Pa_n + Q\frac{a_n - nV^n}{i}$$

One can find $(Ia)_n$ by setting $P = 1$ and $Q = 1$, and $(Da)_n$ by setting $P = n$ and $Q = -1$.

The TI BA II Plus™ makes this very simple and formula free if n is not too large. Suppose, for example, that $i = 0.06$ and we wish to find $(Ia)_5$. This is the present value of the cash flow sequence {1,2,3,4,5} and can be easily computed by going to the CF option...
and entering the values 1,2,3,4,5 for C01, …, C05.\footnote{Note that CF₀ = 0 in this calculation.} This is summarized in the following table

<table>
<thead>
<tr>
<th>Time</th>
<th>Received</th>
<th>Cashflow</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>CF₀</td>
<td>NA</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>C01</td>
<td>F01=1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>C02</td>
<td>F02=1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>C03</td>
<td>F03=1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>C04</td>
<td>F04=1</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>C05</td>
<td>F05=1</td>
</tr>
</tbody>
</table>

The present value can be found by computing the NPV with I = 6.

The answer displayed is 12.14691247.

To find \( (Dₐ)₅ \) entering the values 5,4,3,2,1 for C01, …, C05. (Each value has a frequency of 1, which is the default.) The present value can be found by computing the NPV with I = 6. The answer displayed is 13.12727204.

The TI BA II Plus™ will allow a sequence of as many as 24 cash flows. Thus computations of \( (Iₐ)ₙ \) for n>24 require knowledge of the basic formula.

The practical reason for using this notation and the underlying theory is to find the present value of cash flow sequences like:

\[
650, 700, 750, 800, 850, 900, 950, 1000
\]

This can be done directly using the above formula in P and Q. For small sequences like the one above direct use of the calculator CF option is faster. We will illustrate this in the next section.
4) Sample problems for calculator application

In this section we will illustrate some basic problems which can be solved directly on the TI BA II Plus™. Actuarial examination problems tend to be more involved, and we will wait until the next section to illustrate those.

Unknown rate of interest

You are going to invest $3000 now to receive a payment of $3000 in 2 years and a payment of $4000 in 4 years. What is your effective interest rate?

Solution. This is an IRR problem for the CF option. The cashflows are summarized in the following table.

<table>
<thead>
<tr>
<th>Paid</th>
<th>Time</th>
<th>Received</th>
<th>Cashflow</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3,000</td>
<td>←</td>
<td>0</td>
<td>CF₀</td>
<td>NA</td>
</tr>
<tr>
<td>1</td>
<td>→</td>
<td>0</td>
<td>C0₁</td>
<td>F₀₁=1</td>
</tr>
<tr>
<td>2</td>
<td>→</td>
<td>3000</td>
<td>C₀₂</td>
<td>F₀₂=1</td>
</tr>
<tr>
<td>3</td>
<td>→</td>
<td>0</td>
<td>C₀₃</td>
<td>F₀₃=1</td>
</tr>
<tr>
<td>4</td>
<td>→</td>
<td>4000</td>
<td>C₀₄</td>
<td>F₀₄=1</td>
</tr>
</tbody>
</table>

Begin by choosing the CF option and clearing any prior work. The keystrokes are

**CF**

**2nd CLR WORK**

Enter the cashflows CF₀ = -3000, C₀₁ = 0, C₀₂ = 3000, C₀₃ = 0, C₀₄ = 4000. Then compute the IRR.

**IRR CPT**

The answer is 32.60112138%.

This problem was originally designed to require the student to set up the basic IRR equation and solve it. The equation turns out to be a quadratic, which makes it solvable by a student who must use a limited calculator on an exam. The CF option is quicker.
**Additional payment problem**

You planned to accumulate $100,000 in 20 years by making equal deposits at the start of each year in a fund earning 7% per annum. After 9 years you decide to add an amount X to each deposit in the last 11 years so that the final accumulation would be $120,000. Find a) the original deposit and b) the additional amount X.

**Solution** The payments are made at the start of the year, so the calculator must be set to BGN mode. You should also clear TVM. The beginning keystrokes are

\[ \text{2nd CLR TVM} \]
\[ \text{2nd BGN (BGN is displayed above the PMT key)} \]
\[ \text{2nd SET (to choose the BGN mode)} \]
\[ \text{CE/C} \]

a) You wish to accumulate \( FV = 100,000 \) in \( N = 20 \) years at a rate of \( I/Y = 7 \). Compute the payment, \( PMT \), based on the above values. The keystrokes are

\[ \begin{align*}
20 & \quad \text{N} \\
7 & \quad \text{I/Y} \\
100000 & \quad \text{FV} \\
\text{CPT} & \quad \text{PMT}
\end{align*} \]

The initial regular deposit was \( PMT = -2,279.71 \).\(^{12}\)

b) With \( N = 11 \) payments left you now wish to accumulate an additional \( FV = 20,000 \) by making the additional payment \( X \). Enter the new \( N \) and \( FV \) and compute \( PMT \). The keystrokes are

\[ \begin{align*}
11 & \quad \text{N} \\
20,000 & \quad \text{FV} \\
\text{CPT} & \quad \text{PMT}
\end{align*} \]

The additional amount \( X \) is \(-1,184.24\).

\(^{12}\) We will report the value computed by the calculator with its sign. Multiple choice exams typically give the numbers calculated as positive numbers without the sign.
Retirement fund problem

You wish to accumulate a fund for retirement by depositing $1200 at the beginning of each year for 20 years. At the end of year 20 you plan to start making 10 equal annual withdrawals, with the first withdrawal immediately. Find the amount of the annual withdrawal if the effective rate of interest is 6% during the first 20 years and 5% thereafter.

Solution First note that all payments and withdrawals are made at the beginning of the year. The calculator must be set to BGN mode and TVM cleared as in the last problem.

The problem has two basic components.

1) How much money do you have at the end of 20 years to fund the pension?

   The funding is done using a PMT = -1200 for N = 20 years at a rate of I/Y = 6. Enter these values and compute FV = 46,791.27. The keystrokes are

   \[
   \begin{align*}
   & \text{20} \quad \text{N} \\
   & \text{6} \quad \text{I/Y} \\
   & \text{-1200} \quad \text{PMT} \\
   & \text{CPT} \quad \text{FV}
   \end{align*}
   \]

   The amount accumulated in the fund is FV = 46,791.27.

2) What payment (withdrawal) can be made from that fund starting a time 20?

   At the end of year 20 the fund has a present value PV = 46,791.27. After that time it will earn at a rate of I/Y = 5. You wish to make N = 10 withdrawals so that at the end of that time the fund is used up and FV=0. Enter those values, and compute the withdrawal PMT. The keystrokes are

   \[
   \begin{align*}
   & \text{0} \quad \text{FV} \\
   & \text{10} \quad \text{N} \\
   & \text{5} \quad \text{I/Y} \\
   & \text{46791.27} \quad \text{PV} \\
   & \text{CPT} \quad \text{PMT}
   \end{align*}
   \]

   The answer is PMT = -5771.13.

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Current value problem

Three years ago an annuity was set up to pay you $1000 at the end of each quarter for 16 years. You have just received payment number 12. Find the current value of the annuity using a nominal rate of 8% per year.

Solution. The current value is the sum of two components –the future value of past payments and the present value of future payments. The payments are quarterly at an interest rate of \( I/Y = 2 \) per quarter for a total of 64 quarters. Enter the rate –and be sure to check that you have reset the calculator to END mode.

1) Future value of past payments. \( N = 12 \) payments of 1000 = PMT have already been received. Enter these values and compute \( FV = -13,412.09 \).

2) Present value of future payments. \( N = 52 \) payments remain in the future, and will result in a final \( FV = 0 \). Enter these values and compute \( PV = -32,144.95 \).

The final answer (made positive) is \( 13,412.09 + 32,144.95 = 45,557.04 \).

Final payment problem

A loan of $9,000 is to be repaid by making payments of $1,000 to begin at the end of this year and continue as long as necessary. The final payment is to be made when the remaining balance is less than $1,000 and will be larger than the regular payments because it includes the remaining balance. The annual rate of interest is 8.25%. Find the time and amount of the final payment.

Solution. Clear TVM.

a) Time of final payment. The loan is for an amount of \( PV = 9000 \). It requires interest of \( I/Y = 8.25 \). It is paid off with a payment of \( PMT = -1000 \). To find the time to pay off the loan, compute \( N = 17.11 \). This tells us that the loan is closest to final payment at time \( N = 17 \). That is when the final payment will be made.

b) Amount of final payment. To find the unpaid balance at time \( N=17 \), immediately after the last payment of $1000 is made, set \( N = 17 \) and compute \( FV \), the remaining obligation. The answer is \( FV = -109.70 \). The total final payment consists of the base payment of 1,000 and the unpaid balance of 109.70. The final payment at time \( N = 17 \) is 1109.70.

---

13 In this problem and many subsequent problems we will save time and increase our focus on the central problem by not giving the explicit keystrokes. We will assume that by now the reader knows the keystrokes to set \( PMT = 1000 \).
Annuity yield for two annuities with the same present value.

One annuity pays 5 at the end of each year for 40 years. Another pays 6 at the end of each year for 20 years. The present values are equal at an effective rate of interest i. Find i.

Solution. This problem is designed for the student to write down the annuity formulas for the two annuities, set them equal and then solve a quadratic equation. It is designed to require knowledge of first principles and not admit a calculator solution.

Regardless of design, the CF option will solve the problem. If the present values of the two annuities are equal at the rate i, then the present value of the difference of the two annuities is 0 at the rate i. The difference of the two annuities is -1 = 5 – 6 for the first 20 years, and 5 for the last 20 years.

<table>
<thead>
<tr>
<th>Years</th>
<th>Annuity 1</th>
<th>Annuity 2</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-20</td>
<td>5</td>
<td>6</td>
<td>-1</td>
</tr>
<tr>
<td>21-40</td>
<td>5</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>PV at i</td>
<td>x</td>
<td>x</td>
<td>0</td>
</tr>
</tbody>
</table>

If the present value of the difference cash flows is 0 at the rate i, then i is the IRR of that sequence. We can find i by going to the CF option and entering the sequence C01 = -1, F01 =20, C02 = 5, F02 = 20, and then computing the IRR. The answer is 8.38%.

Increasing annuity problem.

An annuity immediate has annual end-of-year payments of 600, 650, 700, ...,1000. Find the present value at the rate i=.06.

Solution. This problem was designed to be solved using increasing annuity formulas. However, there are only nine cashflows:

600, 650, 700, 750, 800, 850, 900, 950, 1000.

It is a simple matter to enter the nine values as C01, …, C09 and find the NPV using I = 6. The reader should recall that the ENTER key must be used for each cashflow entry, and that after the cashflows are all entered the sequence of keystrokes and displays is

\[
\begin{align*}
\text{NPV} \\
I = 6 \\ \text{ENTER} \\
\downarrow \\
\text{NPV} = \text{CPT}
\end{align*}
\]

The answer is NPV = 5,309.85.
Loan amortization

A loan of 20,000 is being repaid with equal quarterly payments at the end of each quarter for 5 years. The interest rate is 5% convertible quarterly. Find the outstanding loan balance at the end of 2.5 years and the total principal and interest paid during that time.

Solution. The loan has a PV of 20,000 and an interest rate per quarter of $\frac{9}{4} = 1.25 = I/Y$ for $N = 20$ quarters. Payments are made at END of period. The payment is not given and must be found first. Enter the values above and compute PMT = -1136.41. There are now two ways to find the balance, principal paid and interest.

Amortization option solution. Key in 2nd AMORT to begin the amortization option. There are $N = 10$ periods in 2.5 years, so we are amortizing from period 1 to period 10. Enter the values

$P1 = 1, \ P2 = 10$

and scroll down. The required results are shown as

BAL = 10,620.33
PRN = -9,379.67
INT = -1,984.41

Direct TVM solution. Quit the amortization option by keying 2nd QUIT. The balance at time $N = 10$ is given by FV. Enter 10 for N and compute FV.

Balance = FV = 10,620.33.

The amount of principal paid in the first 10 periods is the original loan amount less the balance at time 10.

Principal paid = 20,000 – 10,620.33 = 9,379.67.

The interest paid is the total paid less the principal paid. The total paid is the payment multiplied by 10. The interest paid is the total paid less principal paid.

Interest paid = 10·1136.41 – 9379.67 = 1,984.43.

The AMORT solution above was 1984.41. The slight difference is due to rounding. The calculator has in memory a payment of -1,136.407793. In practice one would pay 1,136.41, not the more exact amount. This small discrepancy will not lead to problems on student examinations, but does cause some confusion in practice, as we have discussed on page 10..
Bond price and amortization

a) A 10 year $1000 bond bearing a 7% coupon rate payable semiannually is bought to yield 5% semiannually. The bond is redeemable at par. Find the price.

b) Find the interest and principal recognized in the fourth payment.

c) Find the total of the interest paid column and the principal paid column in the bond amortization schedule.

d) Suppose the bond in a) could be purchased for 950. What would its yield be?

Solution.

a) $N = 20$ semiannual coupon payments are made. The coupon payment is $0.07\times1000/2 = 35 = PMT$. The redemption at par means the $FV = 1000$. The bond is purchased to yield $5/2 = 2.5 = I/Y$ per period. Enter these values in TVM and compute the price $PV$. The answer is $-1,155.89$.

b) Since the bond is issued at a premium of 155.89, part of that premium must be amortized and recognized as a payment of principal in each period. Select the AMORT option and enter $P1=P2 = 4$. Then scroll down to review answers. The amortization of premium recognized as a principal payment in the fourth payment is $PRN = 6.57$. The interest recognized is $INT = 28.43$.

c) To review the entire amortization of premium, scroll to the display line for $P1$. Enter the value 1 for $P1$ and 20 for $P2$. Scroll down to review answers, and you will see that the amortization option does not include the final payment of $FV = 1000$. The displays show successively that

$$
\begin{align*}
  BAL &= -1,000 \\
  PRN &= 155.89 \\
  INT &= 544.11
\end{align*}
$$

The total principal paid is $115.89 + 1000 = 1155.89$, the price paid for the bond. The total interest paid is 544.11.

d) Return to TVM. If the bond were purchased for 950, the value of $PV$ would change to -950. Enter that value for $PV$ and compute $I/Y$. The answer is displayed as $I/Y = 3.863474796$. To obtain the nominal annual yield, multiply by 2. The yield quoted for this price would be $7.726949592$. 

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5) Actuarial examination problems with calculator solutions. In this section we will review recent Part 2 actuarial examination problems which can be solved in a fairly direct manner with a financial calculator.

May 2000, Problem 24

A small business takes out a loan of 12,000 at a nominal rate of 12%, compounded quarterly, to help finance its start-up costs. Payments of 750 are made at the end of every 6 months for as long as is necessary to pay back the loan.

Three months before the 9th payment is due, the company refinances the loan at a nominal rate of 9%, compounded monthly. Under the refinanced loan, payments of \( R \) are to be made monthly, with the first monthly payment to be made at the same time that the 9th payment under the old loan was to be made. A total of 30 monthly payments will completely pay off the loan.

Determine \( R \).

(A) 448  
(B) 452  
(C) 456  
(D) 461  
(E) 465

Solution.

Original loan. In any refinancing situation, the first task is to find the status of the original loan when the refinancing takes place. There is one preliminary step. The original loan is complicated by the difference in the quarterly compounding period and the semi-annual payment period. We must first determine the semi-annual rate that actually applies to the semi-annual payments. This is found by compounding the periodic rate of 3% (12%/4):

\[
1.03^2 - 1 = .0609
\]

The loan is refinanced after the 8th payment. We first use TVM to find the balance immediately after the 8th payment by entering the values

\[
PV = 12,000 \quad PMT = -750 \quad I/Y = 6.09 \quad N = 8
\]

and computing \( FV = -11,809.35 \). This is the balance at time 8.

The loan is refinanced three months (exactly one quarter later.) Thus the original loan is in force for one additional quarter after the 8th payment is made. At a quarterly interest rate of 3%, the balance on the original loan at time of refinancing is

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11,809.35(1.03) = 12,163.63

The refinanced loan. The amount of the new loan is 12,163.63. The new loan will be monthly and have a nominal rate of 0.75%. However payments will not start immediately. A sequence of 30 monthly payments to pay off the new loan will start at the time that was scheduled for the 9th payment of the original loan. A table is helpful to sort out the confusion.

<table>
<thead>
<tr>
<th>Month after payment 8 on original loan</th>
<th>Loan in force</th>
<th>Loan Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Original</td>
<td>11,809.35</td>
</tr>
<tr>
<td>1</td>
<td>Original</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Original</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Original</td>
<td>11,809.35(1.03) = 12,163.63</td>
</tr>
<tr>
<td>4</td>
<td>Refinanced – accruing</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Refinanced – accruing</td>
<td>12,183.63(1.0075)^2 = 12,346.77</td>
</tr>
<tr>
<td>6</td>
<td>Refinanced – payments begin</td>
<td></td>
</tr>
</tbody>
</table>

The first of the monthly payments on the new loan of 30 months will be made at month 6 in the table above. Since loans are paid on an immediate annuity basis (end of month payments), time 6 in the above table is time 1 for our new loan. The starting point (or time 0) for the refinanced loan is month 5 above. At that time the unpaid balance has increased for two months at 0.75% interest, to reach the amount of 12,346.77 shown above.

With this information, we can find the required payment on the new 30 month loan. Enter the TVM values

\[ PV = 12,346.77 \quad N = 30 \quad I/Y = .75 \]

and compute PMT. The answer is 461.13, which is choice D.

**A note on May 2000, Problem 26**

This problem requires the calculation of \((D_a)_{\bar{n}}\) for \(n = 20\). That can be found in less than a minute by entering the cash flows 20, 19, \ldots, 1 into the CF option and finding NPV. Actuarial students must know the formula for this, but the CF option provides a useful check.
May 2000, Problem 29

A firm has proposed the following restructuring for one of its 1000 par value bonds. The bond presently has 10 years remaining until maturity. The coupon rate on the existing bond is 6.75% per annum paid semiannually. The current nominal semiannual yield on the bond is 7.40%.

The company proposes suspending coupon payments for four years with the suspended coupon payments being repaid, with accrued interest, when the bond comes due. Accrued interest is calculated using a nominal semiannual rate of 7.40%.

Calculate the market value of the restructured bond.

(A) 755
(B) 805
(C) 855
(D) 905
(E) 955

Solution. This examination problem was designed to have a simple solution, based on a basic insight. If bond payments are suspended but repaid with all due accrued interest, the bond has not changed because all accrued obligations will be paid. The value of the bond is exactly the same as if the payments had never been suspended. We can find the original value easily. Set N=20, PMT = 33.75 (that is 6.75%/2 of 1000), I/Y = 3.7 and FV = 1000. Compute PV to find the value before payments were suspended. The rounded answer is 954.64, which corresponds to choice E.

It is often difficult to rapidly see the insight intended by a question writer. We will now give a solution that is more likely to occur to an ordinary mortal.

The bond has semi-annual coupon payments of .0675(1000)/2 = 33.75 and was originally designed to pay 1,000 at maturity. The restructured bond will be paid off over the same remaining N = 20 periods, but the payments will change. The restructured bond cash flows are summarized in the table below.

<table>
<thead>
<tr>
<th>Periods</th>
<th>Coupon paid</th>
<th>Original redemption</th>
<th>Deferred coupon</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-8</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>9-19</td>
<td>33.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>33.75</td>
<td>1,000</td>
<td>FV of deferred coupons</td>
</tr>
</tbody>
</table>

To completely describe the restructured bond, we need to find the future value of the deferred coupons 20 periods from now. All valuation in this problem is done at a semiannual interest rate of 7.40%/2 = 3.7%. It is simpler to first find the present value of the deferred coupons. There are N = 8 deferred coupons with PMT = 33.75 to be valued using the TI BA II Plus™
at an interest rate of I/Y = 3.7. Enter these values in TVM and compute the present value. The value today is PV = -230.07. The future value of these deferred coupons in 20 periods is

$$230.07(1.037)^{20} = 475.82.$$ 

Now that the payments on the restructured bond are known, we can create a table showing how to use the CF option to find the present value of that bond.

<table>
<thead>
<tr>
<th>Time</th>
<th>Received</th>
<th>Cashflow</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>CF0</td>
<td>NA</td>
</tr>
<tr>
<td>1-8</td>
<td>0</td>
<td>C01</td>
<td>F01=8</td>
</tr>
<tr>
<td>9-19</td>
<td>33.75</td>
<td>C02</td>
<td>F02=11</td>
</tr>
<tr>
<td>20</td>
<td>1,509.57</td>
<td>C03</td>
<td>F03=1</td>
</tr>
</tbody>
</table>

The final payment is the sum of the last coupon payment, the original 1,000 redemption and the value of the deferred coupons, 33.75 + 1,000 + 475.82.

Enter the cash flows with their frequencies and compute the NPV at I = 3.7. The answer is displayed as NPV = 956.64, which is choice E.

The present value computed using CF could also be done using the TVM option, but CF is probably a bit quicker.
May 2000, Problem 39

Sally lends 10,000 to Tim. Tim agrees to pay back the loan over 5 years with monthly payments payable at the end of each month.

Sally can reinvest the monthly payments from Tim in a savings account paying interest at 6%, compounded monthly. The yield rate earned on Sally’s investment over the five-year period turned out to be 7.45%, compounded semi-annually.

What nominal rate of interest, compounded monthly, did Sally charge Tim on the loan?

(A) 8.53%
(B) 8.59%
(C) 8.68%
(D) 8.80%
(E) 9.16%

Solution. We will begin by finding the amount that Sally had in her savings account at the end of 5 years. The 5 years contain 20 semi-annual periods. We are told that her semiannual yield rate was 7.45%/2 = 3.725%. If she originally invested 10,000 and earned 3.725% in 20 periods, the ending amount in her account must be

\[
10,000 \times (1.03725)^{10} = 14,415.66.
\]

Sally got to this final amount by depositing Tim’s monthly payments into an account earning 6%/12 = 0.5%. There were 60 payments in the 5 years. We can find the payment by using TVM and entering the values

\[
N = 60 \quad I/Y = .5 \quad FV = 14,415.66 \quad PV = 0
\]

and computing PMT = -206.62. That is the payment that Tim made.

Now we can compute the rate of interest on the loan, since we know the loan amount (10,000), payment and number of periods. Enter the TVM values

\[
FV = 0 \quad PV = 10,000.
\]

The correct values for PMT and N are already stored. Compute I/Y = .733376599, the monthly rate. To obtain the quoted nominal yield, multiply the monthly rate by 12. The answer is 8.80, which is choice D.
May 2000, Problem 43

A 1000 par value 5-year bond with 8.0% semiannual coupons was bought to yield 7.5% convertible semiannually.

Determine the amount of premium amortized in the 6th coupon payment.

(A) 2.00
(B) 2.08
(C) 2.15
(D) 2.25
(E) 2.34

Solution.

N = 10 semiannual coupon payments are made. The coupon payment is 
.08·1000/2 = 40 =PMT. The redemption at par means the FV = 1000. The bond is purchased to yield 7.5/2 = 3.75 = I/Y per period. Enter these values in TVM and compute the price PV. The answer is -1,020.53.

Since the bond is issued at a premium of 20.53, part of that premium must be amortized and recognized as a payment of principal in each period. Select the AMORT option and enter P1=P2 = 6. Then scroll down to review answers. The amortization of premium recognized as a principal payment in the fourth payment is PRN = 2.08. This corresponds to choice B.
May 2000, Problem 47

Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985.

Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200.

How much did Jim accumulate in his fund on December 31, 1999?

(A) 53,572
(B) 53,715
(C) 53,840
(D) 53,966
(E) 54,184

Solution We can use the CF option to get the present value of Jim’s payments. The monthly interest rate is 6%/5 = 0.5%. The payments are summarized in the following table.

<table>
<thead>
<tr>
<th>Time</th>
<th>Begin month</th>
<th>Payment</th>
<th>Cashflow</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>200</td>
<td>CF&lt;sub&gt;0&lt;/sub&gt;</td>
<td>NA</td>
</tr>
<tr>
<td>1-58</td>
<td>2-59</td>
<td>200</td>
<td>C01</td>
<td>F01=58</td>
</tr>
<tr>
<td>59-71</td>
<td>60-72</td>
<td>0</td>
<td>C02</td>
<td>F02=13</td>
</tr>
<tr>
<td>72-179</td>
<td>73-180</td>
<td>200</td>
<td>C03</td>
<td>F03=108</td>
</tr>
</tbody>
</table>

Since the payments are made at the beginning of a month, we include an immediate payment of 200 for \( CF_0 \). Subsequent payments in CF are mathematically interpreted as end of period payments, but the end of period 1 is the beginning of period 2, and so on.

Enter the cash flows with their frequencies and compute the NPV at \( i = 0.5 \). The answer is displayed as \( NPV = 21,938.78605 \). This is a present value. The future value in 180 months is

\[
21,938.78605 (1.005)^{180} = 53,839.83.
\]

This answer corresponds to choice C. The problem could also be solved in steps using TVM.

Using the TI BA II Plus™

Hassett and Stewart

34
November 2000, Problem 12

Kevin takes out a 10-year loan of \(L\), which he repays by the amortization method at an annual effective interest rate of \(i\). Kevin makes payments of 1000 at the end of each year.

The total amount of interest repaid during the life of the loan is also equal to \(L\). Calculate the amount of interest repaid during the first year of the loan.

(A) 725
(B) 750
(C) 755
(D) 760
(E) 765

Solution We know the payment (\(PMT = 1000\)) and the term (\(N = 10\)) for this loan. Thus we could use TVM to solve for the interest rate if we knew the loan amount, \(L = PV\). We can find the loan amount using the information given above as follows:

\[
\text{Loan amount} + \text{Interest paid} = \text{Total payments} = 1000 \cdot 10 = 10,000
\]

\[
\text{Loan amount} = \text{Interest paid} = L
\]

\[
2L = 10,000 \quad \rightarrow L = 5000 = PV
\]

To find the loan interest rate \(i\), clear TVM and enter the values

\[
PMT = -1000 \quad N = 10 \quad PV = 5000.
\]

Compute \(I/Y\). The answer is \(I/Y = 15.09841448\).

The interest paid in the first year is

\[
L \cdot i = 5,000 \cdot (0.1509841448) = 754.92. \text{ This corresponds to choice C.}^{14}
\]

---

\(^{14}\) The interest could also be found using the AMORT feature.
November 2000, Problem 22

Jerry will make deposits of 450 at the end of each quarter for 10 years. At the end of 15 years, Jerry will use the fund to make annual payments of \( Y \) at the beginning of each year for 4 years, after which the fund is exhausted. The annual effective rate of interest is 7%.
Determine \( Y \).

(A) 9573  
(B) 9673  
(C) 9773  
(D) 9873  
(E) 9973

Solution. The annual effective rate is 7% in all years, but since payments are made quarterly during the first 10 years, we need to convert the annual effective rate to a quartely rate:

\[
(1.07)^{1/4} - 1 = .017058525.
\]

Note that in the first 10 years payments are made at end of quarter, but in the final 4 years payments are at the beginning of the year.

Calculation of the amount accumulated in the fund in 10 years. Clear TVM and enter the values

\[ \text{PMT} = -450 \quad \text{N} = 40 \quad \text{I/Y} = 1.7058525. \]

Then compute \( \text{FV} = 25,513.23 \). This is the value of the fund in 10 years.

Value of the fund in 15 years. During the next 5 years no deposits are made to the fund, which grows at 7% per year. The accumulated value after the next 5 years is

\[
25,513.23 \times (1.07)^5 = 35,783.63
\]

Calculation of the unknown payment \( Y \). The payment \( Y \) is made at the beginning of each year for \( N = 4 \) years, with interest rate \( I/Y = 7 \). The beginning value of the account is the accumulated value of the fund, \( PV = 35,783.63 \). Set the mode to BGN and clear TVM. Then enter the above values for \( PV \), \( N \) and \( I/Y \) and compute \( \text{PMT} \). The answer is -9,873.21, which is choice D.

It is wise to reset the mode to END after this problem is finished, since most problems are done in that mode.\(^{15}\)

\(^{15}\) The side trip to BGN mode can be avoided if one cleverly notes that the same answer is obtained by inflating the payments for only 4 years and keeping the calculator in end mode. The author who is typing this footnote is not that clever.
November 2000, Problem 34

An investor took out a loan of 150,000 at 8% compounded quarterly, to be repaid over 10 years with quarterly payments of 5483.36 at the end of each quarter.

After 12 payments, the interest rate dropped to 6% compounded quarterly. The new quarterly payment dropped to 5134.62.

After 20 payments in total, the interest rate on the loan increased to 7% compounded quarterly. The investor decided to make an additional payment of $X$ at the time of his 20th payment. After the additional payment was made, the new quarterly payment was calculated to be 4265.73, payable for five more years.

Determine $X$.

(A) 11,047  
(B) 13,369  
(C) 16,691  
(D) 20,152  
(E) 23,614

Solution. We will denote $B_t$ the unpaid balance owed by the investor at time $t$, immediately after the payment for period $t$ is made. The loan can be divided into 3 stages. We summarize the investor’s situation in the following table.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Quarters</th>
<th>Loan amount</th>
<th>Quarterly rate</th>
<th>Payment</th>
<th>Loan term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-12</td>
<td>150,000</td>
<td>2%</td>
<td>5483.36</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>13-20</td>
<td>$B_{12}$</td>
<td>1.5%</td>
<td>5134.62</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>21-40</td>
<td>$B_{20} - X$</td>
<td>1.75%</td>
<td>4265.73</td>
<td>20</td>
</tr>
</tbody>
</table>

Initially the investor set up a loan whose payment was calculated using TVM with $N = 40$, $I/Y = 2$ and $PV = 150,000$. Enter those values and compute PMT, and you will see that the payment of 5483.36 is correct.

The lender had the right to change the rate at certain points in time. Immediately after the 12th payment was made, the quarterly rate was changed to 1.5%. The balance on the loan at that time can be found in TVM by changing $N$ to 12 and computing FV. The balance $B_{12}$ is displayed as $FV = -116,692.92$.

In stage 2 the investor effectively has a new loan of $116,692.92$ for 28 quarters at 1.5%. You can check the payment given. Enter the values

$PV = 116,692.92$  
$N = 28$  
$I/Y = 1.5$.

Compute the payment and you will see the display $PMT = -5,134.61$. This checks.
On the stage 2 loan, 8 payments are made before the rate changes again. The 8th payment on the stage 2 loan is the 20th payment made in total. Immediately after that 20th payment is made the lender changes the quarterly rate to 1.75%. This is where stage 3 begins.

The new stage 3 loan amount would normally be the outstanding balance on the stage 2 loan after 8 payments. We can find this using TVM. Change the value of N to 8 and compute FV. The answer is displayed as FV = -88,154.33. This is the outstanding balance for the investor immediately after the 20th total payment is made. \( B_{20} = 88,154.33 \).

The investor now pays the additional principal \( X \) to lower the amount of the stage 3 loan to \( B_{20} - X \) and obtain a lower payment.

Recall that the amount of a loan is equal to the present value of its future payments. The stage 3 loan has PMT = -4265.73, I/Y = 1.75, FV = 0 and N = 20. Enter these values in TVM and compute PV. The answer is displayed as PV = 71,463.27, the final stage 3 loan amount. Thus

\[
71,463.27 = B_{20} - X = 88,154.33 - X
\]

\[
X = 16,691.06.
\]

This corresponds to choice C.
November 2000, Problem 38

Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price inflates 4% per year.

To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits an additional $X$ at the beginning of years 4, 5, and 6 to meet his goal.

The annual effective interest rate is 10%.

Calculate $X$.

(A) 7.4  
(B) 7.9  
(C) 8.4  
(D) 8.9  
(E) 9.4

Solution. We first calculate how much money Chuck needs in 10 years. At an inflation rate of 4%, he would need

$$200 \times (1.04)^{10} = 296.05.$$ 

Chuck is depositing money into an account that earns 10% interest. He will only make deposits for 6 years, so by the end of the 6th year he needs to accumulate

$$296.05 / 1.10^4 = 202.21.$$ 

The deposits of 20 are made at the beginning of the year. To see what they will accumulate to in 6 years, clear TVM, set the calculator to BGN and enter the values

PMT = -20  
I/Y = 10  
N = 6. 

Compute FV = 169.74. This accumulation is short of the needed amount by

$$202.21 - 169.74 = 32.47.$$ 

Thus the 3 payments of $X$ must accumulate to 32.47. Enter the new TVM values

N = 3  
FV = 32.47.

Then compute the desired payment, PMT = -8.92. This corresponds to choice D.
November 2000, Problem 48

A 12-year loan of 8000 is to be repaid with payments to the lender of 800 at the end of each year and deposits of X at the end of each year into a sinking fund. Interest on the loan is charged at an 8% annual effective rate. The sinking fund annual effective interest rate is 4%.

Calculate X.

(A) 298
(B) 330
(C) 361
(D) 385
(E) 411

Solution. The sinking fund is necessary because the payment of 800 is not sufficient to pay off the loan. There will be a balance due on the loan in 12 years. To find this balance, clear TVM and set the mode to END. Enter the values

\[ N = 12 \quad \text{PMT} = -800 \quad \text{I/Y} = 8 \quad \text{PV} = 8000. \]

Then compute \( FV = -4,963.66. \)

This balance must be accumulated in the sinking fund with interest rate \( \text{I/Y} = 4 \) by making the payment X for \( N = 12 \) years. Enter the new values

\[ \text{I/Y} = 4 \quad \text{PV} = 0. \]

Compute the payment, \( \text{PMT} = 330.34. \) This corresponds to the choice B.
May 2001, Problem 4

A 20-year loan of 20,000 may be repaid under the following two methods:
i) amortization method with equal annual payments at an annual effective rate of 6.5%
ii) sinking fund method in which the lender receives an annual effective rate of 8% and the sinking fund earns an annual effective rate of $j$
Both methods require a payment of $X$ to be made at the end of each year for 20 years. Calculate $j$.

(A) $j \leq 6.5\%$
(B) $6.5\% < j \leq 8.0\%$
(C) $8.0\% < j \leq 10.0\%$
(D) $10.0\% < j \leq 12.0\%$
(E) $j > 12.0\%$

Solution.

*Amortization method.* Use TVM to calculate the payment on the loan. Enter

PV = 20,000  I/Y = 6.5  N = 20.

Compute PMT = -1,815.13 =X.

*Sinking fund method.* The borrower must pay 8% interest on 20,000 each year -i.e. 1,600. The problem tells us that his total payment is also $X = 1,815.13$. Thus the borrower has 215.13 left each year to deposit into the sinking fund. This payment scheme must accumulate 20,000 in the fund over 20 years. The unknown is the interest rate $j$ which will do this. Clear TVM and enter the values


Compute the interest rate I/Y = 14.179%. This corresponds to choice E.
May 2001, Problem 7

Seth, Janice, and Lori each borrow 5000 for five years at a nominal interest rate of 12%, compounded semi-annually. Seth has interest accumulated over the five years and pays all the interest and principal in a lump sum at the end of five years. Janice pays interest at the end of every six-month period as it accrues and the principal at the end of five years. Lori repays her loan with 10 level payments at the end of every six-month period. Calculate the total amount of interest paid on all three loans.

(A) 8718
(B) 8728
(C) 8738
(D) 8748
(E) 8758

Solution. Each borrower pays interest of 6% per period for 10 semi-annual periods.

Seth makes no payments until the end of 5 years, and must pay semi-annual interest of 6% compounded 10 times. His total due is

\[ 5000 \times (1.06)^{10} = 8,954.24. \]

Of this total amount, 5000 is principal. Thus the interest paid is 3,954.24.

Janice pays the interest due of 5000(0.06) = 300 every year, but no principal until the end. Her total interest paid is 300·10 = 3000.

Lori makes level amortizing payments. We can find her payment using TVM. Clear TVM and enter the values

\[ PV = 5,000 \quad I/Y = 6 \quad N = 10. \]

Compute the payment, PMT = -679.34. If 10 payments of this amount are made, the total paid is 6,793.40. Of this amount, 5,000 is principal and the interest paid is 1,793.40.

For all three borrowers, the total amount of interest paid is

\[ 3954.24 + 3000 + 1793.40 = 8747.64. \]

This corresponds to choice D.
May 2001, Problem 41

Bill buys a 10-year 1000 par value 6% bond with semi-annual coupons. The price assumes a nominal yield of 6%, compounded semi-annually. As Bill receives each coupon payment, he immediately puts the money into an account earning interest at an annual effective rate of $i$. At the end of 10 years, immediately after Bill receives the final coupon payment and the redemption value of the bond, Bill has earned an annual effective yield of 7% on his investment in the bond. Calculate $i$.

(A) 9.50%
(B) 9.75%
(C) 10.00%
(D) 10.25%
(E) 10.50%

Solution. The first sentence tells us that Bill purchased the bond for 1,000. The bond has a coupon rate of 3% for each semi-annual period and is purchased to yield the same 3%.

If Bill has invested 1,000 and earned an effective yield of 7% over 10 years, his original 1000 should grow to

$$1000 \cdot (1.07)^{10} = 1,967.15.$$ 

At time 10, Bill receives a 1,000 redemption payment. Thus he must have accumulated 967.15 = FV in his account. He did this by depositing coupon payments of -30 = PMT into the account for N = 20 periods and earning interest in the account at an unknown rate. We can use TVM to find this rate. Clear TVM and enter the values

PMT = -30 N = 20 FV = 967.15.

Compute the rate I/Y = 4.75964418. This is the rate per semi-annual period. To find the annual effective rate, we compound.

$$1.0475964418^2 - 1 = .097458305$$

This corresponds to choice B.
November 2001, Problem 27

A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of \( X \) to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life.

Calculate \( X \).

(A) 324.73
(B) 326.89
(C) 328.12
(D) 355.45
(E) 450.65

Solution. We will start by finding the amount needed in the fund on the 65th birthday. The retiree wishes to have 3000 per month of income. Since it costs 1000 to obtain 9.65 of lifetime income, the cost per dollar of lifetime income is

\[
\frac{1000}{9.65} = 103.626943
\]

The cost of 3,000 of lifetime income is

\[
3000 \cdot \frac{1000}{9.65} = 310,880.83
\]

This is the amount that must be in the fund on the retiree’s 65th birthday. It will be accumulated by making deposits at the beginning of each of 300 months in an account earning \((8/12)%\). The unknown deposit can be found using TVM. Clear TVM and set the mode to BGN. Enter the values

\[ I/Y = 8/12 \quad FV = 310,880.83 \quad N = 300. \]

Compute the payment, \( PMT = -324.72 \). This corresponds to answer A.
6 A note on the sign convention for TVM

The underlying equation relating the variables in TVM is

\[ PV + PMT \left( \frac{1 - v^n}{i} \right) + \frac{FV}{(1+i)^n} = 0 \]

It is easy to see that a positive value of PV must be offset by negative values of PMT and/or FV for the equality to hold. This equation leads consistently to solutions in which money paid to you is positive and money paid out is negative.

Another possible version of the TVM equation is

\[ PV = PMT \left( \frac{1 - v^n}{i} \right) + \frac{FV}{(1+i)^n} \]

With this equation the key values all come out positive on most problems. For example, if \( PV = 1000, i = 5\%, N=6 \) and \( FV = 0 \), the calculated value of PMT is 197.02. In a way this seems simpler, since all answers are positive and there is no sign convention to worry about. However, negative values do occur in a surprising way. Consider the problem of finding the annuity values \( a_{\bar{n}|} \) and \( s_{\bar{n}|} \) for \( n = 10 \) and \( i = 6\% \). In both cases \( PMT = 1 \). For the computation of \( a_{\bar{n}|} \) we set \( FV = 0 \) and the formula gives the answer 7.36009, which is positive. For the computation of \( s_{\bar{n}|} \) we set \( PV = 0 \) and the formula gives the negative answer -13.18079. This is a confusing sign convention unless you are constantly thinking about the formula. However, this is how the TI BA 35™ calculator solves the problems above. We find the sign convention on the TI BA II Plus™ calculator easier to deal with because of its consistent interpretation.