Functions with Countably Many Discontinuities Are Riemann Integrable

Since I pretty well muddled the proof that functions with a finite number of discontinuities are Riemann integrable, I will try to make amends by proving that functions with a countable number of discontinuities are Riemann integrable.

Assume that $f$ is defined on $[a, b]$ and $|f(x)| \leq B$. It will be convenient to define $f(x) = f(a)$ for $-\infty < x < a$ and $f(x) = f(b)$ for $b < x < \infty$. Note that when $f$ is continuous at $x = a$ or $x = b$ that will be true for this extension.

Suppose that the discontinuities of $f$ are given by $S = \{x_n, n \in \mathbb{N}\}$. Then, given $\epsilon > 0$, we have the intervals $I_n = \{x : |x-x_n| < 2^{-n}\epsilon\}$. Note the sum of the lengths of the $I_n$'s is $\epsilon$. Consider $\mathcal{O} = \bigcup_{n=1}^{\infty} I_n$. $\mathcal{O}$ is an open set in $\mathbb{R}$, and $J = [a, b] \setminus \mathcal{O}$ is compact.

Since $f$ is continuous at all points of $J$, given $x \in J$, there is a $\delta_x > 0$ such that $M_x - m_x < \epsilon$, where $M_x = \sup\{f(t), t \in I_x\}$ and $m_x = \inf\{f(t), t \in I_x\}$ and $I_x = \{t : x-\delta_x \leq t \leq x+\delta_x\}$. Since $J$ is compact and covered by the open interiors of the $I_x$'s, there is a finite set of $I_x$'s that cover $J$ that I will label as $J_1, ..., J_N$.

Finally, form the partition $P$ consisting the endpoints of $J_1, ..., J_N$ that lie in $[a, b]$ plus $a$ and $b$. Any interval in $P$ is either a subinterval of one of the $I_j$'s or its interior is in $\mathcal{O}$. Since $\mathcal{O}$ is a union of intervals whose lengths sum to $\epsilon$, we have

$$U(f, P) - L(f, P) \leq \epsilon(b-a) + 2B\epsilon,$$

where the first term on the right comes from intervals which are subintervals of one the $I_j$'s and the second comes from intervals with interiors in $\mathcal{O}$. Hence, since $\epsilon$ arbitrary, $f$ is Riemann integrable by the standard criterion.

In writing this I used ideas from “stackexchange.com/questions/263189”