Assignment 4

1. Suppose that $f$ is real-valued, bounded on $[a,b]$ and $f^3$ is Riemann-integrable on $[a,b]$. Does it follow that $f^2$ is Riemann-integrable?

2. Suppose that $f$ is Riemann-integrable on $[c,1]$ for every $c \in (0,1]$. Define the improper Riemann integral of $f$ on $[0,1]$ by

$$\int_0^1 f(x)dx = \lim_{c \downarrow 0} \int_c^1 f(x)dx$$

if this limit exists and is finite.

a) Show that this agrees with the Riemann integral when $f$ is Riemann-integrable on $[0,1]$.

b) Construct an $f$ for which the limit in (2) exists which is not Riemann integrable, and for which the limit in (2) does not exist when $f$ is replaced by $|f|$.

3. (# 4 on Basic F’07, also in a more obscure version # 3 W’06) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $|f''(x)| \leq B$ for all $x$.

a) Prove that

$$|2Af(0) - \int_{-A}^A f(x)dx| \leq 2BA^3/3$$

b) Use the result of part a) to justify the estimate

$$|\int_a^b f(x)dx - \frac{b-a}{n}\sum_{k=1}^n f(a + \frac{2k-1}{2n}(b-a))| \leq \frac{C}{n^2}$$

where $C$ does not depend on $n$.

4. Suppose that $f$ on $[0,1]$ is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/m, & \text{if } x=n/m \text{ is lowest terms.} \end{cases}$$

Show that $f$ is Riemann-integrable on $[0,1]$ and its integral is 0.

5. Build a sequence of closed subintervals of $[0,1]$ as follows: set $I_1 = [0,1]$, $I_2 = [0,1/2]$, $I_3 = [1/2,1]$, $I_4 = [0,1/4]$, $I_5 = [1/4,1/2]$, and so on, filling out $[0,1]$ over and over, and halving the length of the intervals each time you start over. Let $f_n(x) = 1$ on $I_n$ and 0 elsewhere. Show that $\lim_{n \to \infty} \int_0^1 f_n(x)dx = 0$ even though $\lim_{n \to \infty} f_n(x)$ fails to exist for any $x \in [0,1]$.

6. (Rudin) Letting $\{x\}$ denote the fractional part of $x$, i.e. $x$ minus the largest integer less than or equal to $x$ consider $f(x) = \sum_{n=1}^\infty \{nx\}/n^2$. Find the discontinuities of $f$ and show that they form a countable dense set in $\mathbb{R}$. Show that $f$ is nevertheless Riemann-integrable on every bounded interval.