Linearity

Definition (Version 1: the abstract definition)
A function $f: \mathbb{R}^n \to \mathbb{R}^k$ is called a \textit{linear function} if it has the following two properties:

1. $f(c\mathbf{v}) = cf(\mathbf{v})$ for all vectors $\mathbf{v}$ in $\mathbb{R}^n$ (the domain of $f$) and all scalars $c$.
   (In short: “$f$ commutes with scalar multiplication”.)

2. $f(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$ for all vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$.
   (In short: “$f$ commutes with addition”.)
Theorem (Version 2 of definition: formula of a linear function)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a linear function if and only if it has the following form:

$$f \left( \begin{bmatrix} X \\ Y \\ Z \\ \vdots \end{bmatrix} \right) = \begin{bmatrix} _X + _Y + _Z + \cdots \\ _X + _Y + _Z + \cdots \\ \vdots \\ _X + _Y + _Z + \cdots \end{bmatrix}$$

where the _'s represent constants (numbers), and can be any numbers at all.
Linearity

Theorem (Version 3 of definition: linear function as matrix)

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a linear function if and only if it has the following form:

$$f \left( \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \right) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix},$$

or, more concisely:

$$f(\vec{v}) = M\vec{v}$$

where $M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k,1} & a_{k,2} & \cdots & a_{k,n} \end{bmatrix}$. 
Eigenvalues and eigenvectors

Definition
Given an \( n \times n \) matrix \( M \) (or equivalently, a linear function \( f: \mathbb{R}^n \to \mathbb{R}^n \)), an \( n \)-dimensional vector \( \vec{v} \) is called an eigenvector for \( M \) (or for \( f \)) with eigenvalue \( \lambda \) if \( \vec{v} \neq 0 \) and

\[
M \vec{v} = \lambda \vec{v}
\]

(or equivalently \( f(\vec{v}) = \lambda \vec{v} \)).

Things to note:

1. Here \( M \) is a matrix, \( \vec{v} \) is a vector, and \( \lambda \) is a scalar.

2. This definition means that eigenvalues and eigenvectors always go together: for any eigenvector of a matrix, there is a corresponding eigenvalue, and vice-versa.
Diagonal matrices

Definition
Given a square \((n \times n)\) matrix, the \textit{diagonal} of the matrix is the set of entries along the diagonal line going from the top left to the bottom right of the matrix:

\[
\begin{bmatrix}
4 & 2 & 9 \\
3 & 5 & 0 \\
2 & 4 & 2
\end{bmatrix}
\]

A matrix is called a \textit{diagonal matrix} if all of its entries above and below the diagonal are 0.

\[
\begin{bmatrix}
4 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
Bases and coordinates

Definition
A \textit{basis} for $\mathbb{R}^n$ is a list of $n$ vectors \{$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$\} for which the following is true:

\[\text{For any vector } \vec{v} \text{ in } \mathbb{R}^n, \text{ it is possible to write } \vec{v} \text{ as a linear combination of } \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n. \text{ In other words, for any vector } \vec{v} \text{ in } \mathbb{R}^n, \text{ there are some scalars } c_1, c_2, \ldots, c_n \text{ for which} \]

\[\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_n \vec{u}_n.\]

In the expression above, the scalars $c_1, c_2, \ldots, c_n$ are called the \textit{coordinates} of $\vec{v}$ with respect to this basis.

Geometrically, this means that it’s possible to reach any point in $\mathbb{R}^n$ by going some distance parallel to $\vec{u}_1$, followed by some distance parallel to $\vec{u}_2$, followed by some distance parallel to $\vec{u}_3$, etc....