Note about notation: In Exercise 43, the scalars \( c_1 \) and \( c_2 \) were used to refer to the coordinates of a point in the new coordinate system defined by a different basis \( \{ \mathbf{u}, \mathbf{v} \} \). In class, I referred to these as simply “\( R, S \)-coordinates”. For convenience, I will continue to use that terminology/notation here. So remember that the \( c_1 \) from Exercise 43 is \( R \) here, and \( c_2 \) is \( S \).

46. Define a \( 2 \times 2 \) matrix whose columns are the vectors \( \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \) from Exercise 43. As you learned in Tuesday’s lecture, this matrix (called \( T \)) converts \( R, S \)-coordinates into ordinary \( X, Y \)-coordinates, in the following way:

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = T \begin{bmatrix} R \\ S \end{bmatrix}
\]

(1)

Verify this for the point \((2, 3)\) by taking the values of \( R \) and \( S \) (\( c_1 \) and \( c_2 \)) that you found for this point in Exercise 43, and “converting” those \( R, S \)-coordinates back into \( X, Y \)-coordinates. What do you notice? Do this for two of the other points from Exercise 43 as well.

47. (Non-Sage problem) Expand the vector equation (1) into a system of two equations, and solve simultaneously for \( R \) and \( S \) in terms of \( X \) and \( Y \). Once you have done this, rewrite your result in the form of a vector/matrix equation again, that is, in the form

\[
\begin{bmatrix} R \\ S \end{bmatrix} = W \begin{bmatrix} X \\ Y \end{bmatrix}
\]

(2)

for some matrix \( W \). You probably want to do this on paper, but when you’re finished, enter the matrix \( W \) that you found into Sage.

48. Just as the matrix \( T \) converts \( R, S \)-coordinates into \( X, Y \)-coordinates (equation (1)), the matrix \( W \) you found in Exercise 47 converts \( X, Y \)-coordinates into \( R, S \)-coordinates (equation (2)). Use this to find the \( R, S \)-coordinates for the point \((1, -6)\). To check that you’ve found the right \( R, S \)-coordinates, go back to your interactive from Exercise 43, and make \((1, -6)\) the target point. Use the \( R \) and \( S \) that you’ve found here as your \( c_1 \) and \( c_2 \), and see if you hit the target.

49. Repeat the previous exercise for some other point.

50. Since \( W \) does exactly the opposite of what \( T \) does, the \( W \) matrix is called the inverse of the \( T \) matrix. Sage can compute the inverse of a matrix quickly and easily. If \( A \) is an \( n \times n \) matrix in Sage, then \( A.invse() \) will give the inverse of \( A \). Use this to compute the inverse of the matrix \( T \) (from Exercise 46). Compare the result to Exercise 47.
51. The matrix \( \mathbf{M} = \begin{bmatrix} 7 & -4 \\ 4 & -3 \end{bmatrix} \) has eigenvectors \( \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \), the same vectors we have been using here. Using Sage or hand calculations, verify that these are the eigenvectors of \( \mathbf{M} \), and find the eigenvalue corresponding to each one.

52. Recall that we now have three linear functions, each with a corresponding matrix:

- The function that converts \( R, S \)-coordinates to \( X, Y \)-coordinates, represented by the matrix \( \mathbf{T} \)
- The function \( f \left( \begin{bmatrix} X \\ Y \end{bmatrix} \right) = \begin{bmatrix} 7X - 4Y \\ 4X - 3Y \end{bmatrix} \), represented by the matrix \( \mathbf{M} \). Note that this function naturally operates on \( X, Y \)-coordinates.
- The function that converts \( X, Y \)-coordinates to \( R, S \)-coordinates, represented by the matrix \( \mathbf{W} \) (the inverse of \( \mathbf{T} \))

To figure out how the function \( f \) operates on \( R, S \)-coordinates, we need to compose these three functions:

\[
\begin{bmatrix} R \\ S \end{bmatrix}_{\text{input}} \xrightarrow{\text{convert coords}} \begin{bmatrix} X \\ Y \end{bmatrix}_{\text{input}} \xrightarrow{f} \begin{bmatrix} X \\ Y \end{bmatrix}_{\text{output}} \xrightarrow{\text{convert back}} \begin{bmatrix} R \\ S \end{bmatrix}_{\text{output}}
\]

Use Sage to compute the matrix of this composition of functions. What do you notice about the result?