Exercise 5.1.2.
Let us let $X_0 = 20$. For $r = 0.5$, we have

\[ X_1 = 0.5 \cdot X_0 = 10, \quad X_2 = 0.5 \cdot X_1 = 5. \]

It shrinks.
- For $r = 1$, pretty clearly $X_1 = 1 \cdot X_0 = 20$ and $X_2 = 20$ too.
- For $r = 1.5$, we have
  \[ X_1 = 1.5 \cdot X_0 = 30, \quad X_2 = 1.5 \cdot X_1 = 45. \]

It grows.

Exercise 5.1.3.
This is a classic problem on the weird nature of exponential growth. Let $X$ denote the proportion of the lake that is covered. We know that this situation is modelled by

\[ X_{t+1} = 2 \cdot X_t \]
as it doubles every day. On day $t = 30$ the lake is totally covered, so $X_{30} = 1$. Then we know that

\[ X_{30} = 2 \cdot X_{29} \implies \frac{X_{30}}{2} = X_{29} \implies \frac{1}{2} = X_{29}. \]

Therefore on the 29th day, the lake will be half-covered and the city should clean it up.

Exercise 5.1.4.
A 10% per year growth rate means that

\[ X_{t+1} = X_t + 0.10 \cdot X_t = 1.1 \cdot X_t. \]

We know that $X_0 = 10$, and we want to find $X_{10}$. Using the general formula that

\[ X_N = r^N \cdot X_0 \]

we can see that $X_{10} = (1.1)^{10} \cdot 10 \approx 26$ rabbits (as we shouldn’t have fractional rabbits). To write a quick Sage script,
X = 10 # set the initial value to 10
for j in range(0,10): # it'll do it 10 times
    X = 1.1 * X
print(X)

Exercise 5.1.5.
If we let $X_t$ be our amount of radioactive material after $t$ half-lives, we know that $X_0$ is however much we start with and $X_1 = 0.5 \cdot X_0$. After another half-life, we should have $X_2 = (0.5) \cdot X_1 = (0.5)^2 \cdot X_0$. That means that the $r$ in our general equation should be 0.5:

$$X_N = (0.5)^N \cdot X_0.$$

This means that $X_{10} = (0.5)^{10} \cdot X_0 = \frac{X_0}{1024}$. Since we are only interested in the fraction of $X_0$ that remains, our answer is just $\frac{1}{1024}$.

Exercise 5.1.6.

(a) Let $X_t$ be our money after $t$ years, so $X_0 = 1000$. Every year we get 2%, so similar to the rabbits above we have

$$X_{t+1} = X_t + 0.02 \cdot X_t = (1.02) \cdot X_t \implies X_N = (1.02)^N \cdot 1000.$$

We just need to compute this number for $N = 5, 10, 20$, which is now a question for your calculator:

$$X_5 = 1104.08, \quad X_{10} = 1218.99, \quad X_{20} = 1485.95.$$

Note: this is rounded to the nearest cent.

(b) We want to find $N$ such that

$$X_N = (1.02)^N \cdot 1000 \geq 10000 \implies (1.02)^N \geq 10.$$

Again, we have reached a calculator problem. You can either be fancy and use logarithms to get an exact answer ($N = 116.27$) or plug in values for $N$ until you find the first one that makes $(1.02)^N \geq 10$. Remember that this is a discrete time model, so we can’t use 116.29 – we have to round up to $N = 117$.

To take advantage of Sage, you can make an interact:

```python
@interact
def interest(N=(1,1000,1)):
    print((1.02)**N)
```

Or you can use some fancier programming:
X = 1000  # our initial value
N = 0  # our initial time
while X < 10000:
    X = X*(1.02)  # do a round of interest
    N += 1  # then add 1 to the year we’re in
print(N)
# the loop ends when we have over 10000 dollars
# so the last step is to print out the year N

Exercise 5.1.7.
This is just a plug-and-chug problem, as my math teacher used to say: we are using
the logistic equation

\[ X_{t+1} = 1.2 \cdot X_t(1 - X_t), \quad X_0 = 0.42. \]

Therefore

\[ X_1 = 1.2 \cdot 0.42 \cdot (1 - 0.42) = 0.29232, \]
\[ X_2 = 1.2 \cdot 0.29232 \cdot (1 - 0.29232) = 0.24824282112. \]

You probably rounded between steps, but don’t do that! You can round when you
write down your answer, but in the model you need to keep it as precise as possible.
Remember that in chaotic systems (which this one might be), little changes can cause
big differences later on.

Exercise 5.2.FE1.
Here’s our lovely table: the characteristics of chaotic behavior are listed on the left
column in the helpful mnemonic BSAD:

<table>
<thead>
<tr>
<th></th>
<th>Exponential growth</th>
<th>Equilibrium behavior</th>
<th>Oscillation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Sensitive*</td>
<td>Yes†</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Aperiodic</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Deterministic</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

*Sensitive dependence on initial conditions, but that doesn’t fit in a nice chart
†To see that exponential growth is sensitive to initial conditions, consider the
(non-discrete) model \(N' = N\). Let us take the initial condition \(N_1(0) = 1\). Then the
solution to this equation is actually \(N_1(t) = e^t\). Suppose we take the initial condition
\(N_2(0) = 1.1\). Then the solution is \(N_2(t) = 1.1 \cdot e^t\). Then the distance between these
trajectories is

\[ N_2(t) - N_1(t) = (1.1 - 1) \cdot e^t = 0.1 \cdot e^t = (N_2(0) - N_1(0)) \cdot e^t. \]

Check on page 235 of Modelling Life to confirm that this matches the definition of
sensitive dependence on initial conditions.
Exercise 5.3.1.
The definition for a discrete-time equilibrium point is
\[ X_{t+1} = X_t. \]
For our purposes, we need to solve
\[ X_{t+1} = r \cdot X_t = X_t \implies r \cdot X_{eq} = X_{eq} \]
Well, there aren’t very many ways to get a solution to this. We have that \((r-1)X_{eq} = 0\), so we must have \(X_{eq} = 0\) (unless \(r = 1\), but that’s not exponential growth). The only equilibrium point is if there is zero life.

Exercise 5.3.2.
To solve this problem, we want to find
\[ X_{eq} = X_{eq} \cdot e^{r \cdot (1 - X_{eq}/K)} \]
Subtracting the lefthand side to the right and factoring, we have
\[ 0 = X_{eq} \cdot (1 - e^{r \cdot (1 - X_{eq}/K)}) \]
This gives us one easy solution: \(X_{eq} = 0\). Otherwise, we have to solve
\[ 1 - e^{r \cdot (1 - X_{eq}/K)} = 0 \implies e^{r \cdot (1 - X_{eq}/K)} = 1 \implies r \cdot (1 - X_{eq}/K) = 0. \]
Since \(r \neq 0\), that means that \(1 - X_{eq}/K = 0\), i.e. \(X_{eq} = K\). That makes sense – you get equilibrium if you manage to land right on the carrying capacity.

Exercise 5.3.3.
This was covered in class, but I’ll repeat it here. We saw above in 5.3.1 that the only equilibrium point of \(X_{t+1} = r \cdot X_t\) is \(X_{eq} = 0\). To consider stability, what happens if we move a little away to \(X_0 = \varepsilon\) (some small number)? Well, if \(r > 1\), then \(X_1 = r \cdot \varepsilon > \varepsilon\). If we keep going, we have that \(X_N = r^N \cdot \varepsilon\) keeps getting bigger and bigger, so that makes the point unstable. However, if \(r < 1\), then \(X_1 = r \cdot \varepsilon < \varepsilon\), so \(X_N = r^N \cdot \varepsilon\) keeps getting smaller and smaller, towards \(X_{eq} = 0\) again.

Exercise 5.3.4.
We need to take the derivative of \(f(X) = X \cdot e^{r \cdot (1 - X/K)}\) then plug in our value \(X_{eq} = K\). The derivative here is
\[
\frac{df}{dX} = X \cdot \frac{\cdot e^{r \cdot (1 - X/K)} + e^{r \cdot (1 - X/K)}}{K} \\
= (1 - rX/K)e^{r \cdot (1 - X/K)}. 
\]
If we plug in \(X_{eq} = K\), we get \(\frac{df}{dX}|_{X=K} = 1 - r\). By the previous exercise, the value of \(r\) such that \(|1 - r| > 1\) is when the equilibrium point is unstable. Since we are assuming that \(r\) is a positive number, \(1 - r > 1\) is always false, so we only need
to find when $1 - r < -1$. Simplifying, we have $r > 2$. Therefore when $r > 2$ the equilibrium is unstable and when $0 < r < 2$ the equilibrium is stable.

**Exercise 5.3.5.**

Looking at the picture, for $r = 3.1$, there are two $X$ values. This means that the model will oscillate (discretely) between two points.

For $r = 3.5$, it looks like there are four $X$ values, so the model will oscillate (discretely) between four points instead of two this time.

Finally, for $r = 3.7$, we’ve hit chaotic behavior. There’s no telling exactly what the model is doing.

**Exercise 6.2.1.**

(a) Add componentwise:

$$
\begin{pmatrix}
1 + (-2) \\
2 + 0 \\
3 + 5
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
2 \\
8
\end{pmatrix}
$$

(b) Scalar multiplication is also performed componentwise:

$$
\begin{pmatrix}
-3 \cdot 4 \\
-3 \cdot 6 \\
-3 \cdot -9
\end{pmatrix}
= 
\begin{pmatrix}
-12 \\
-18 \\
27
\end{pmatrix}
$$

(c) You can’t add vectors of different dimensions, so this is impossible.

(d) Do the same thing as you would in $\mathbb{R}$: do the sum in the parentheses first, then scalar multiply. Or use the distributive property:

$$
5 \cdot \left( \begin{pmatrix}
0 \\
1
\end{pmatrix} + \begin{pmatrix}
7 \\
3
\end{pmatrix} \right) = 
5 \cdot \begin{pmatrix}
7 \\
4
\end{pmatrix} = 
\begin{pmatrix}
35 \\
20
\end{pmatrix}
$$

(e) Multiply then add, same order of operations as usual:

$$
-4 \cdot \begin{pmatrix}
1 \\
0
\end{pmatrix} + 2 \cdot \begin{pmatrix}
0 \\
1
\end{pmatrix} = 
\begin{pmatrix}
-4 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
2
\end{pmatrix} = 
\begin{pmatrix}
-4 \\
2
\end{pmatrix}
$$

(f) Same as (e), but with three terms:

$$
5 \cdot \begin{pmatrix}
1 \\
0
\end{pmatrix} - 3 \cdot \begin{pmatrix}
0 \\
1
\end{pmatrix} + 8 \cdot \begin{pmatrix}
0 \\
0
\end{pmatrix} = 
\begin{pmatrix}
5 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
-3
\end{pmatrix} + \begin{pmatrix}
0 \\
8
\end{pmatrix} = 
\begin{pmatrix}
5 \\
-3
\end{pmatrix}
$$
Exercise 6.2.2.
It’s all the options of putting a 1 together with three 0s:
\[ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

Exercise 6.2.3.
If the 1 is in position 5, then it must be \( \vec{e}_5 \). See the above exercise in the case of \( \mathbb{R}^4 \).

Exercise 6.2.4.
All these problems work in the same way: separate the vector into only top and only bottom parts, then factor out a scalar multiple.

(a) \[
\begin{pmatrix} 45 \\ 12 \end{pmatrix} = \begin{pmatrix} 45 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = 45 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 12 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 45 \vec{e}_1 + 12 \vec{e}_2
\]

(b) \[
\begin{pmatrix} 387 \\ 509 \end{pmatrix} = \begin{pmatrix} 387 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 509 \end{pmatrix} = 387 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 509 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 387 \vec{e}_1 + 509 \vec{e}_2
\]

(c) \[
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \cdot \vec{e}_1 + b \cdot \vec{e}_2
\]

Exercise 6.2.5.

(a) Yes, this is a linear combination of \( a \) and \( b \).
(b) \( e^X \) is not a linear function, so this is not a linear combination.
(c) \( t^2 \) is not a linear function (it’s quadratic), so this is not a linear combination.
(d) This is tempting, but the constant term 5 makes this also not a linear combination. If it were \(-6X + 4W\), then it would be a linear combination of \( X \) and \( W \).

Exercise 6.2.6.
To make a smoothie, you put in a certain amount of a finite list of ingredients. To use an example from real life\(^1\), a smoothie is 2 bananas, 2 cups of milk, 1/2 cup of peanut butter, 2 tablespoons of honey, and 2 cups of ice cubes. Then we could represent the smoothie as
\[
2B + 2M + 0.5P + 2H + 2I.
\]

Implicit here are that the variables have units attached to them. What are they?

---

\(^1\)https://www.allrecipes.com/recipe/221261/peanut-butter-banana-smoothie/
Exercise 6.2.7.
If any component of the function is not give a linear combination of $X, Y, Z$, then the function isn’t linear. You can then prove it using the definition.

(a) We see that $X^2$ is not a linear combination of $X$ and $Y$, so let’s find a problem.
Consider the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then we expect that
\[ 2 \cdot f \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = f \left( 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \]
However,
\[ 2 \cdot f \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 2 \cdot \begin{pmatrix} 1^2 \\ 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]
but
\[ f \left( 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 2^2 \\ 2 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \]
Therefore $f$ is not linear.

(b) $\sqrt{X}$ is also not a linear function, so we just need to prove it fails the definition. Again, we can use scalar multiplication to test it. Let $X = 1$ and take our scalar as 2. Then
\[ 2 \cdot f(1) = 2 \cdot \sqrt{1} \neq \sqrt{2 \cdot 1} = f(2 \cdot 1). \]

(c) This would be linear but for the term $XY$, which is not a linear combination of $X$ and $Y$ but their product. Let us take the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the scalar 2:
\[ 2 \cdot f \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 2 \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \]
However,
\[ f \left( 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = f \left( \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \]
and so we don’t have the equality we need and $f$ can’t be linear.

(d) This one finally does look linear, as each of the function components is a linear combination of $X, Y, Z$. Unfortunately, we have to prove it. The most rigorous proof is the following: consider two arbitrary vectors $\vec{v}_1 = \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$
and \( \vec{v}_2 = \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \) and let \( c \in \mathbb{R} \) be a scalar. We can actually prove both properties of linearity at once if we show that

\[
f(\vec{v}_1 + c \cdot \vec{v}_2) = f(\vec{v}_1) + c \cdot f(\vec{v}_2).
\]

This separates into the two separate conditions from the book if you let \( c = 1 \) or \( \vec{v}_1 = \vec{0} \).

For us, we will compute the lefthand side and the righthand side separately and show they’re equal.

\[
f(\vec{v}_1 + c \cdot \vec{v}_2) = f \left( \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + c \cdot \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} \right) = f \begin{pmatrix} X_1 + c \cdot X_2 \\ Y_1 + c \cdot Y_2 \\ Z_1 + c \cdot Z_2 \end{pmatrix} = \begin{pmatrix} 2(X_1 + c \cdot X_2) \\ 4(Y_1 + c \cdot Y_2) \\ 3(Z_1 + c \cdot Z_2) \end{pmatrix}
\]

We’ll leave it there for now. For the righthand side,

\[
f(\vec{v}_1) + c \cdot f(\vec{v}_2) = f \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + c \cdot f \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 2X_1 \\ 4Y_1 \\ 3Z_1 \end{pmatrix} + c \cdot \begin{pmatrix} 2X_2 \\ 4Y_2 \\ 3Z_2 \end{pmatrix} = \begin{pmatrix} 2X_1 + c \cdot 2X_2 \\ 4Y_1 + c \cdot 4Y_2 \\ 3Z_1 + c \cdot 3Z_2 \end{pmatrix}
\]

But this is exactly the same as what we got for the lefthand side with we factor out the 2, 4, and 3 from each component. This proves (at last) that \( f \) is linear.

Exercise 6.2.8.
This is the property of commuting with scalar multiplication, as while \( \vec{X} \in \mathbb{R}^1 \) we are thinking of as a (1-dimensional) vector, \( X \in \mathbb{R} \) is just a real number, so we can pull it outside of \( f \).
Exercise 6.2.9.
We have to use both additivity and commuting with scalar multiplication here.

Exercise 6.2.10.
We write an arbitrary vector in $\mathbb{R}^3$ as \[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}.
\]
The first step is to write
\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} = \begin{pmatrix}
X \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
Y \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
Z
\end{pmatrix} = X \cdot \vec{e}_1 + Y \cdot \vec{e}_2 + Z \cdot \vec{e}_3
\]
Therefore
\[
f \left( \begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} \right) = f(X \cdot \vec{e}_1 + Y \cdot \vec{e}_2 + Z \cdot \vec{e}_3) = X \cdot f(\vec{e}_1) + Y \cdot f(\vec{e}_2) + Z \cdot f(\vec{e}_3).
\]
We then assign the values $f(\vec{e}_1) = a$, $f(\vec{e}_2) = b$, and $f(\vec{e}_3) = c$, where $a, b, c \in \mathbb{R}$ are just numbers. We can conclude that
\[
f \left( \begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix} \right) = aX + bY + cZ.
\]