Midterm
Version A

Last Name: ____________________________________________

First Name: ____________________________________________

Student ID: _____________________________________________

Signature: _____________________________________________

Lab Section: 1A (Tue 8–10) 1B (Tue 10–12) 1C (Tue 12–2)
1D (Tue 2–4) 1E (Wed 10–12) 1F (Wed 12–2)
1G (Wed 2–4) 1H (Wed 4–6) 1I (Thu 8–10)
1J (Thu 10–12) 1K (Thu 12–2) 1L (Thu 2–4)
1M (Thu 4–6) 1N (Fri 10–12) 1O (Fri 12–2)
1P (Fri 2–4)

Instructions: Do not open this exam until instructed to do so. You will have 90 minutes to complete the exam. Please print your name and student ID number above, and circle your lab section. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure your phone is silenced and stowed with your other belongings at the front of the room. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.

Please do not write below this line.

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1. The New York City polio epidemic of 1916 has been modeled by considering the number of susceptible individuals \((S)\), the number of infected individuals \((I)\), and the number of individuals who have recovered from infection \((R)\). Consider a discrete-time model for this epidemic, based on the following assumptions:

- Each day, on average, 14% of susceptible individuals become infected.
- 11% of infected individuals die each day.
- 6% of infected individuals recover each day.
- Most individuals who have recovered have some immunity against further infection, but it does not always last forever. So approximately 0.4% of recovered individuals become susceptible again each day.

(a) (8 points) What is the matrix \(M\) for this discrete-time model? (Use the order \(S, I, R\) for your variables and equations.)

(b) (2 points) Suppose that you wanted to simulate this model one week at a time, rather than one day at a time. What matrix would you use? (You don’t need to compute this matrix. Just write an expression for it in terms of your answer to part (a).)
2. Emperor penguins can be subdivided into pre-breeder (\(P\)), breeding adults (\(B\)), and non-breeding adults (\(A\)). Consider a discrete-time model for these populations, based on the following assumptions:

- Each breeding adult gives birth to, on average, 0.4 young per year.
- The young pre-breeder grow to breeding age in an average of 5 years.
- Each year approximately 5% of breeding adults become too old to breed (and thus become non-breeding adults).
- Pre-breeder have a per-capita death rate of 18% per year.
- Breeding adults have a per-capita death rate of 4% per year.
- Non-breeding adults have a per-capita death rate of 8% per year.

(a) (8 points) What is the matrix for this discrete-time model? (Use the order \(P, B, A\) for your variables and equations.)

(b) (2 points) If there are initially 50 pre-breeder, 300 breeding adults, and 75 non-breeding adults, how many of each will there be a year later?
3. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear function.
   
   (a) (3 points) If $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector of $f$ with eigenvalue $3$, and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $f$ with eigenvalue $-2$, what is $f \left( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right)$? What is $f \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$?

   (b) (3 points) Write $\begin{bmatrix} 8 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. That is, find scalars $a$ and $b$ for which
   
   $\begin{bmatrix} 8 \\ -5 \end{bmatrix} = a \begin{bmatrix} 3 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
(c) (5 points) Using your answers to parts (a) and (b), what is \( f \left( \begin{bmatrix} 8 \\ -5 \end{bmatrix} \right) \)?

(Hint: Also use the definition of a linear function. If you were unable to answer (a) or (b), for partial credit here do as much as you can without filling in the actual values from the previous parts.)
4. Circle true or false for each of the following. **Give a brief explanation/justification of each answer.** (For the first two, your explanation should mention the key technical term(s)/phrase(s) that are most closely related.)

(a) (3 points) In a system with chaotic behavior, we can precisely predict everything about the long term future of the system.

TRUE    FALSE

(b) (3 points) In a system with chaotic behavior, we cannot predict anything about the long term future of the system.

TRUE    FALSE

(c) (3 points) Chaos only occurs in very complex models where you take into account many factors (i.e., many state variables).

TRUE    FALSE
5. Let $M$ be the matrix
\[
\begin{bmatrix}
-1 & -6 & -9 \\
-6 & -13 & -21 \\
4 & 10 & 16
\end{bmatrix}
\]

(a) (3 points) One of the eigenvectors of $M$ is
\[
\begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}
\]
What is its corresponding eigenvalue?

(b) (5 points) Another eigenvalue of $M$ is $\lambda = -1$. Find a corresponding eigenvector for it.
6. Suppose that $f$ and $g$ are linear functions with the following properties:

\[
\begin{align*}
&f \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \quad f \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \text{and} \quad f \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\
&g \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -4 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad g \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}.
\end{align*}
\]

(a) (5 points) Explain whether or not $f \circ g$ makes sense, and whether or not $g \circ f$ makes sense. For each of these ($f \circ g$ and $g \circ f$), if it exists, what is its domain and codomain? (That is, describe it as function: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$.) Note: Your explanation here should not use matrices at all; explain this strictly in terms of functions.

(b) (5 points) Compute the matrix of either $f \circ g$ or $g \circ f$. 