1. (10 points) In order to better understand the complex feedback relationship between global warming and the health of tropical forests, you are studying the movement of carbon through a tropical forest ecosystem. To keep your model simple, you’ll track the mass of carbon in plants \((P)\), the mass of carbon in animals \((A)\), and the mass of carbon in bacteria \((B)\). Your research has determined the following:

- Each year, 31% of the carbon in plants passes to animals that eat the plants, and 12% of it goes to bacteria that decompose dead plant matter. The rest remains in the plants.
- Each year, 18% of the carbon in animals goes to bacteria, and 4% of it escapes into the atmosphere. The rest remains in the animals.
- Each year, 37% of the carbon in the bacteria leaves the ecosystem (via the atmosphere, groundwater, etc). 26% of the carbon in bacteria is transferred into the animals, and 22% into the plants. The rest remains in the bacteria.

Write a discrete-time model for this dynamical system. (Your final answer should be in a form like \([\text{next state}] = M[\text{current state}]\).)
Simplify:
\[
\begin{align*}
    P_{t+1} &= P_t - 0.43 P_t + 0.22 B_t = 0.57 P_t + 0.22 B_t \\
    A_{t+1} &= 0.31 P_t + A_t - 0.22 A_t + 0.26 B_t = 0.31 P_t + 0.78 A_t + 0.26 B_t \\
    B_{t+1} &= 0.12 P_t + 0.18 A_t + B_t - 0.85 B_t = 0.12 P_t + 0.18 A_t + 0.15 B_t
\end{align*}
\]

Matrix Form:
\[
\begin{bmatrix}
P_{t+1} \\
A_{t+1} \\
B_{t+1}
\end{bmatrix} =
\begin{bmatrix}
0.57 & 0 & 0.22 \\
0.31 & 0.78 & 0.26 \\
0.12 & 0.18 & 0.15
\end{bmatrix}
\begin{bmatrix}
P_t \\
A_t \\
B_t
\end{bmatrix}
\]
2. You are tracking the spread of rabies through a population of red foxes in Western Australia. Your model, which simply tracks the number of healthy and infected foxes, is based on the following assumptions:

- The healthy foxes have a per-capita birth rate of 31% per year, and a per-capita death rate of 17% per year.
- The infected animals do not reproduce, but die at a per-capita rate of 38% per year.
- When a healthy fox encounters an infected one, there is some chance that the healthy one will become infected. So the per-capita rate at which healthy foxes become infected is proportional to the number of infected foxes, with a proportionality constant of 0.07.

(a) (7 points) Write down a differential equation (continuous time) model for this system, based on these assumptions.

\[
\begin{align*}
H &= \# \text{ of healthy foxes} \\
I &= \# \text{ of infected foxes}
\end{align*}
\]

\[
\begin{align*}
\dot{H} &= 0.31H - 0.17H - 0.07HI \\
\dot{I} &= 0.07HI - 0.38I
\end{align*}
\]

or

\[
\begin{bmatrix}
\dot{H} \\
\dot{I}
\end{bmatrix} =
\begin{bmatrix}
0.14H - 0.07HI \\
0.07HI - 0.38I
\end{bmatrix}
\]

Question 2 continues on the next page...
(b) (2 points) Is this model linear? If so, what is its matrix? If not, why not?

No, it is not linear, due to the presence of the $0.07H\cdot I$ terms.

For a model to be linear, every term must have the form $(\text{constant}) \cdot (\text{variable})$, and the whole equation can only have multiple terms like this added together.

(c) (3 points) Would you expect this model to have chaotic behavior? Why or why not?

No, it cannot have chaotic behavior, because it is an ordinary differential equation (ODE) model, with only 2 variables. ODEs must have at least 3 variables in order for chaotic behavior to occur.

(This is a consequence of the Poincaré–Bendixson Theorem.)
3. The following is a two-stage model of the population of feral horses in Storey County, Nevada, in which $F$ represents the number of fillies (young female horses) and $M$ represents the number of mares (female horses of breeding age).

\[
\begin{align*}
F_{n+1} &= 0.35F_n + 0.25M_n \\
M_{n+1} &= 0.45F_n + 0.95M_n
\end{align*}
\]

(a) (5 points) Write the matrix of this model, and find its eigenvalues.

Matrix: \[
\begin{bmatrix}
0.35 & 0.25 \\
0.45 & 0.95
\end{bmatrix}
\]

Eigenvalues: \[
\begin{align*}
\Gamma &= 0.35 + 0.95 = 1.3 \\
\Delta &= 0.35 \cdot 0.95 - 0.25 \cdot 0.45 = 0.22
\end{align*}
\]

\[
\lambda^2 - 1.3 \lambda + 0.22 = 0
\]

\[
\lambda = \frac{1.3 \pm \sqrt{1.3^2 - 4 \cdot 0.22}}{2} = \frac{1.3 \pm \sqrt{0.81}}{2}
\]

\[
\lambda = 1.1 \quad \text{or} \quad \lambda = 0.2
\]

(b) (8 points) For each of the eigenvalues you just computed, find a corresponding eigenvector. (Hint: After you write out the initial equations, you can multiply everything by 100 to get rid of all the decimals and work only with whole numbers.)

For $\lambda = 1.1$:

\[
\begin{bmatrix}
0.35 & 0.25 \\
0.45 & 0.95
\end{bmatrix} \begin{bmatrix}
F \\
M
\end{bmatrix} = 1.1 \begin{bmatrix}
F \\
M
\end{bmatrix} \iff \begin{align*}
0.35F + 0.25M &= 1.1F \\
0.45F + 0.95M &= 1.1M
\end{align*}
\]

Question 3 continues on the next page...
Question 3 continued...

\[
\begin{align*}
\begin{cases}
35F + 25M &= 110F \\
45F + 95M &= 110M
\end{cases} &\rightarrow 25M = 75F &\rightarrow M = 3F \\
\begin{cases}
45F + 95M &= 110M \Rightarrow \quad 45F = 15M &\rightarrow 3F = M
\end{cases}
\end{align*}
\]

\[
\text{Eigenvector for } \lambda = 1.1: \begin{bmatrix} 1 \\ 3 \end{bmatrix} \] (or any scalar multiple of this)

For \( \lambda = 0.2 \):

\[
\begin{bmatrix}
0.35 & 0.25 \\
0.45 & 0.95
\end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix} = 0.2 \begin{bmatrix} F \\ M \end{bmatrix}
\]

\[
\begin{cases}
0.35F + 0.25M = 0.2F \\
0.45F + 0.95M = 0.2M
\end{cases} \Rightarrow \text{Multiply everything by 100...}
\]

\[
\begin{align*}
\begin{cases}
35F + 25M &= 20F \\
45F + 95M &= 20M
\end{cases} &\rightarrow 25M = -15F &\rightarrow 5M = -3F \\
45F + 75M &= 0 &\rightarrow 3F + 5M = 0 &\text{same}
\end{align*}
\]

\[
\text{Eigenvector for } \lambda = 0.2: \begin{bmatrix} 5 \\ -3 \end{bmatrix} \] (or any scalar multiple of this)
4. You and your friend are studying a difference equation whose behavior is chaotic. You each simulate this equation for many iterations, but you round the initial conditions to 3 decimal places, whereas your friend uses 4 decimal places.

(a) (5 points) How similar or different do you expect the data from the two simulations to be? Explain. (Be sure to mention the key concept/property that is relevant here.)

At first, they should be quite close to each other, but they will get farther and farther apart until eventually the numbers look completely different. The reason for this is sensitive dependence on initial conditions, which says that, over time, the distance between the two simulations will grow exponentially. Since the distance between them starts out very small (in the 4th decimal place) and exponential growth of a small value starts slowly, they will be close at first, but will eventually diverge completely.

(b) (5 points) In what way will the data from the two simulations be similar, even in the long term? Again, explain. (For full credit, mention the key concept/property that is relevant.)

Due to the fact that chaotic behavior approaches a strange attractor (or chaotic attractor), the two simulations will still show the same overall pattern of behavior, even in the long run. For example, average values of state variables, distribution of points in the state space, and short-term patterns/trends will be the same between the two simulations.
5. Suppose you know that \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) is a linear function, and that

\[
f \left( \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \text{ and } f \left( \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}.
\]

(a) (6 points) What is \( f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right) \)? What is \( f \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \)?

Since \( \begin{bmatrix} 3 \\ -12 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \),

\[
f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right) = f \left( -3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right) = -3 \cdot f \left( \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right) = -3 \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -15 \\ -6 \end{bmatrix}
\]

Since \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} \),

\[
f \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = f \left( \frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) = \frac{1}{2} \cdot f \left( \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}
\]

(b) (4 points) What is \( f \left( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right) \)? (Hint: How is the vector \( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \) related to the two vectors \( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) from part (a)?)

Since \( \begin{bmatrix} 4 \\ -9 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \),

\[
f \left( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right) = f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right) + f \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} 9 \\ -15 \\ -6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 10 \\ -15 \\ -8 \end{bmatrix}
\]
6. Suppose you know that \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a linear function, and that
\[
f \left( \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad f \left( \begin{bmatrix} 2 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}.
\]

(a) (6 points) What is \( f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right) \)? What is \( f \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \)?
\[
\begin{align*}
  f(\begin{bmatrix} 3 \\ -12 \end{bmatrix}) &= f(-3 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix}) = -3 \cdot f(\begin{bmatrix} -1 \\ 4 \end{bmatrix}) = -3 \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -15 \\ -9 \end{bmatrix}, \\
  f(\begin{bmatrix} 1 \\ 3 \end{bmatrix}) &= f(\frac{1}{2} \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}) = \frac{1}{2} \cdot f(\begin{bmatrix} 2 \\ 6 \end{bmatrix}) = \frac{1}{2} \cdot \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.
\end{align*}
\]

(b) (4 points) What is \( f \left( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right) \)? (Hint: How is the vector \( \begin{bmatrix} -4 \\ -9 \end{bmatrix} \) related to the two vectors \( \begin{bmatrix} -3 \\ -12 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) from part (a)?)
\[
\begin{align*}
  f(\begin{bmatrix} 4 \\ -9 \end{bmatrix}) &= f(\begin{bmatrix} 3 \\ -12 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}) = f(\begin{bmatrix} 3 \\ -12 \end{bmatrix}) + f(\begin{bmatrix} 1 \\ 3 \end{bmatrix}) = \begin{bmatrix} 6 \\ -15 \\ -9 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -15 \\ -10 \end{bmatrix}.
\end{align*}
\]
6. Suppose $f$ is a function defined by

$$f \left( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} X - 2Z \\ Y + 4Z \\ 2X - 4Y - 3Z \\ -3X - Y \end{bmatrix}$$

and $g$ is a linear function for which the following is true:

$$g \left( \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} \quad \text{and} \quad g \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}.$$ $g: \mathbb{R}^2 \to \mathbb{R}^3$

(a) (5 points) Does the composition $g \circ f$ exist? If not, explain (in terms of functions, not in terms of matrices) why not. If it does exist, what is the matrix of $g \circ f$? \(\text{No.} \)

$g \circ f (\vec{v})$ means $g(f(\vec{v}))$, meaning that you're plugging the output of $f$ into the input of $g$. But the output of $f$ is a 4-dimensional vector, whereas the input that $g$ expects is 2-dimensional. So this does not work.

Diagram:

```
\begin{array}{c}
\mathbb{R}^3 \rightarrow \mathbb{R}^4 \neq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \\
\end{array}
```

Question 6 continues on the next page...
Question 6 continued...

(b) (5 points) Does the composition \( f \circ g \) exist? If not, explain (in terms of functions, \textit{not} in terms of matrices) why not. If it does exist, what is the matrix of \( f \circ g \)?

Yes: \[ \begin{array}{ccc}
\mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\
\longrightarrow & \quad f \\
\mathbb{R}^3 & \longrightarrow & \mathbb{R}^4
\end{array} \]

Matrix of \( f \): \[
\begin{bmatrix}
1 & 0 & -2 \\
2 & 1 & 4 \\
-3 & -1 & 0
\end{bmatrix}
\]

Since \( g \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \), \( g \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \), \( g \left( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \).

So matrix of \( g \) is \[
\begin{bmatrix}
-4 & -1 \\
2 & 2 \\
5 & 3
\end{bmatrix}
\]

Matrix of \( f \circ g \) is

\[
\begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 4 \\
2 & -4 & -3 \\
-3 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
-2 & 2 \\
2 & -1 \\
5 & 3
\end{bmatrix}
= \begin{bmatrix}
-12 & -4 \\
22 & 11 \\
-27 & -1 \\
4 & -5
\end{bmatrix}
\]
3. Suppose $f$ is a function defined by

$$f\left(\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) = \begin{bmatrix} 2X - Z \\ Y + 3Z \\ 3X - 4Y - 2Z \\ -X - Y \end{bmatrix}$$

and $g$ is a linear function for which the following is true:

$$g\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}.$$

(a) (5 points) Does the composition $f \circ g$ exist? If not, explain (in terms of functions, not in terms of matrices) why not. If it does exist, what is the matrix of $f \circ g$?

Matrix of $f$: \[
\begin{bmatrix}
2 & 0 & -1 \\
0 & 1 & 3 \\
3 & -4 & -2 \\
-1 & -1 & 0
\end{bmatrix}
\]

Matrix of $g$: \[
\begin{bmatrix}
-2 & 2 \\
2 & -1 \\
5 & 3
\end{bmatrix}
\]

Matrix of $f \circ g$: \[
\begin{bmatrix}
2 & 0 & -1 \\
0 & 1 & 3 \\
3 & -4 & -2 \\
-1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
-2 & 2 \\
2 & -1 \\
5 & 3
\end{bmatrix} = \begin{bmatrix}
-9 & 1 \\
17 & 8 \\
-24 & 4 \\
0 & -1
\end{bmatrix}
\]