Midterm 2
Version A

Last Name: __________________________
First Name: __________________________
Last six digits of UID: ________________
Lab Section: (circle one)

1A (TA: Max, Tue 12–2)  
1B (TA: Max, Tue 2–4)  
1C (TA: Jeong, Wed 8–10)  
1D (TA: Jeong, Wed 10–12)  
1E (TA: Daniel, Wed 12–2)  
1F (TA: Daniel, Wed 2–4)  
1G (TA: Ian, Thu 12–2)  
1H (TA: Ian, Thu 2–4)  
1I (TA: Soroush, Fri 8–10)  
1J (TA: Soroush, Fri 10–12)  
1K (TA: Neil, Fri 12–2)  
1L (TA: Neil, Fri 2–4)

2A (TA: Vlad, Tue 2–4)  
2B (TA: Tianran, Wed 8–10)  
2C (TA: Kuan-Ting, Wed 10–12)  
2D (TA: Maithili, Wed 12–2)  
2E (TA: Tianran, Wed 2–4)  
2F (TA: Ali, Thu 8–10)  
2G (TA: Ali, Thu 10–12)  
2H (TA: Kuan-Ting, Thu 2–4)  
2I (TA: Maithili, Fri 8–10)  
2J (TA: Alex, Fri 10–12)  
2K (TA: Vlad, Fri 12–2)  
2L (TA: Alex, Fri 2–4)

By signing below, you affirm that you have neither given nor received unauthorized help on this exam.

Signature: __________________________

Instructions: Do not open this exam until instructed to do so. You will have 90 minutes to complete the exam. Please print your name and the last six digits of your student ID number above. Also print the last six digits of your student ID number on each page of the exam. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure your phone is silenced and stowed with your other belongings at the front of the room. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.
1. (10 points) In order to better understand the complex feedback relationship between global warming and the health of tropical forests, you are studying the movement of carbon through a tropical forest ecosystem. To keep your model simple, you’ll track the mass of carbon in plants \((P)\), the mass of carbon in animals \((A)\), and the mass of carbon in bacteria \((B)\). Your research has determined the following:

- Each year, 31% of the carbon in plants passes to animals that eat the plants, and 12% of it goes to bacteria that decompose dead plant matter. The rest remains in the plants.
- Each year, 18% of the carbon in animals goes to bacteria, and 4% of it escapes into the atmosphere. The rest remains in the animals.
- Each year, 37% of the carbon in the bacteria leaves the ecosystem (via the atmosphere, groundwater, etc). 26% of the carbon in bacteria is transferred into the animals, and 22% into the plants. The rest remains in the bacteria.

Write a discrete-time model for this dynamical system. (Your final answer should be in a form like \([\text{next state}] = M[\text{current state}]\).)
Question 1 continued... Last six digits of UID: ____________
2. You are tracking the spread of rabies through a population of red foxes in Western Australia. Your model, which simply tracks the number of healthy and infected foxes, is based on the following assumptions:

- The healthy foxes have a per-capita birth rate of 31% per year, and a per-capita death rate of 17% per year.
- The infected animals do not reproduce, but die at a per-capita rate of 38% per year.
- When a healthy fox encounters an infected one, there is some chance that the healthy one will become infected. So the per-capita rate at which healthy foxes become infected is proportional to the number of infected foxes, with a proportionality constant of 0.07.

(a) (7 points) Write down a **differential equation** (continuous time) model for this system, based on these assumptions.
(b) (2 points) Is this model linear? If so, what is its matrix? If not, why not?

(c) (3 points) Would you expect this model to have chaotic behavior? Why or why not?
3. The following is a two-stage model of the population of feral horses in Storey County, Nevada, in which $F$ represents the number of fillies (young female horses) and $M$ represents the number of mares (female horses of breeding age).

\[
\begin{align*}
F_{n+1} &= 0.35F_n + 0.25M_n \\
M_{n+1} &= 0.45F_n + 0.95M_n
\end{align*}
\]

(a) (5 points) Write the matrix of this model, and find its eigenvalues.

(b) (8 points) For each of the eigenvalues you just computed, find a corresponding eigenvector. (*Hint: After you write out the initial equations, you can multiply everything by 100 to get rid of all the decimals and work only with whole numbers.*)
Question 3 continued... Last six digits of UID: ____________
4. You and your friend are studying a difference equation whose behavior is chaotic. You each simulate this equation for many iterations, but you round the initial conditions to 3 decimal places, whereas your friend uses 4 decimal places.

(a) (5 points) How similar or different do you expect the data from the two simulations to be? Explain. (Be sure to mention the key concept/property that is relevant here.)

(b) (5 points) In what way will the data from the two simulations be similar, even in the long term? Again, explain. (For full credit, mention the key concept/property that is relevant.)
5. Suppose you know that \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is a linear function, and that

\[
\begin{align*}
\begin{bmatrix}
-3 \\
5 \\
2
\end{bmatrix} \\
\begin{bmatrix}
2 \\
0 \\
-4
\end{bmatrix}
\end{align*}
\]

(a) (6 points) What is \( f \left( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right) \)? What is \( f \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \)?

(b) (4 points) What is \( f \left( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \right) \)? (Hint: How is the vector \( \begin{bmatrix} 4 \\ -9 \end{bmatrix} \) related to the two vectors \( \begin{bmatrix} 3 \\ -12 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) from part (a)?)
6. Suppose $f$ is a function defined by

$$f \left( \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \right) = \begin{bmatrix} X - 2Z \\ Y + 4Z \\ 2X - 4Y - 3Z \\ -3X - Y \end{bmatrix}$$

and $g$ is a linear function for which the following is true:

$$g \left( \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} \quad \text{and} \quad g \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}.$$ 

(a) (5 points) Does the composition $g \circ f$ exist? If not, explain (in terms of functions, not in terms of matrices) why not. If it does exist, what is the matrix of $g \circ f$?
(b) (5 points) Does the composition $f \circ g$ exist? If not, explain (in terms of functions, not in terms of matrices) why not. If it does exist, what is the matrix of $f \circ g$?
Some useful formulas, etc:

The characteristic equation of the matrix

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

is \( \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \).

The quadratic formula says that the roots of \( ax^2 + bx + c = 0 \) are

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]