Exercise 6.5.5 Calculate the eigenvalues and eigenvectors of this matrix with increased birth rate, and use them to explain the behavior in Figure 6.31.

Solution:

We have:

\[
\begin{bmatrix}
0.1 & 2 \\
0.4 & 0.2
\end{bmatrix}
\]

So, the characteristic equation is:

\[
\lambda^2 - (0.1 + 0.2) \cdot \lambda + (0.1 \cdot 0.2 - 0.4 \cdot 2) = 0
\]

Thus,

\[
\lambda^2 - 0.3 \cdot \lambda - 0.78 = 0
\]

The solutions to the equation is:

\[
\lambda_{1,2} = \frac{1}{100} \cdot \left(15 \pm 5\sqrt{321}\right)
\]

\[
\rightarrow \lambda_1 = -0.75 \text{ and } \lambda_2 = 1.05
\]

\[\lambda_1 = -0.75:\]

To find the eigenvector, we have: 
\[0.1 \cdot x + 2 \cdot y = \lambda_1 \cdot x \rightarrow y = -0.423 \cdot x\]

Thus, the eigenvector is:

\[
\begin{bmatrix}
1 \\
-0.423
\end{bmatrix}
\]

The line drawn out by this eigenvector is drawn in red on the graph below.

Thus, along this red line, you know that the trajectory will:

- \[\lambda_1 < 0 \rightarrow \text{oscillating along the line}\]
- \[|\lambda_1| = 0.75 < 1 \rightarrow \text{decreasing in amplitude \rightarrow going towards the origin located on the line}\]
$\lambda_2 = 1.05$: 

To find the eigenvector, we have: $0.1 \cdot x + 2 \cdot y = \lambda_2 \cdot x \rightarrow y = 0.473 \cdot x$

Thus, the eigenvector is:

\[
\begin{bmatrix}
1 \\
0.473
\end{bmatrix}
\]

The line drawn out by this eigenvector is drawn in green on the graph below.

Thus, along this green line, you know that the trajectory will:

- $\lambda_2 > 0 \rightarrow$ non-oscillating along the line
- $|\lambda_2| = 1.05 > 1 \rightarrow$ decreasing in amplitude $\rightarrow$ going away from the origin located on the line

---

**Exercise 6.5.6** Use SageMath to calculate the eigenvalues of $L$. Verify that they are

$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

What do the eigenvalues tell you about the behavior you have just seen? Relate each of the phenomena you saw above to specific properties of the eigenvalues.

(2 Points)

Solution:

We have

\[
L = \begin{bmatrix}
0 & 0 & 1000 \\
0.02 & 0 & 0 \\
0 & 0.05 & 0
\end{bmatrix}
\]

Putting this into cocalc.com to get the eigenvalues:

```python
>> L = matrix(RDF, [[0, 0, 1000], [0.02, 0, 0], [0, 0.05, 0]])
>> L.eigenvalues()
```
We get:
\[-0.499999999999998 + 0.8660254037844385\text{i}, -0.499999999999998 - 0.8660254037844385\text{i}, 0.000000000000002\]

Which agrees with the given eigenvalues.

The absolute value of all eigenvalues is 1 in this 3 dimensional model/system. \(\lambda_1 = 1\): Would mean that there is a plane of equilibrium points, where if you are on the plane, the value of vector from one state to the next will remain the same. \(\lambda_2\) and \(\lambda_3\): The absolute value of these eigenvalues are 1, which would mean that the populations will remain the same size in magnitude (absolute value) in long run. And due to the imaginary part, the populations will oscillate neutrally on another plane.

Further Exercises 6.5

2. A blofish population consists of juveniles and adults. Each year, 50% of juveniles become adults and 10% die. Adults have a 75% chance of surviving from one year to the next and have, on average, four offspring a year.

(2 Points)

a) Write a discrete-time matrix model describing this population:

Solution:

\[
J_{t+1} = 0.4J_t + 4A_t \\
A_{t+1} = 0.5J_t + 0.75A_t
\]

\[
\begin{bmatrix}
J_{t+1} \\
A_{t+1}
\end{bmatrix} = 
\begin{bmatrix}
0.4 & 4 \\
0.5 & 0.75
\end{bmatrix}
\begin{bmatrix}
J_t \\
A_t
\end{bmatrix}
\]

b) If the population this year consists of 50 juveniles and 35 adults, what will next year’s population be?

Solution:

\[
\begin{bmatrix}
J_{t+1} \\
A_{t+1}
\end{bmatrix} = 
\begin{bmatrix}
0.4 & 4 \\
0.5 & 0.75
\end{bmatrix}
\begin{bmatrix}
50 \\
35
\end{bmatrix} = 
\begin{bmatrix}
20 + 140 \\
25 + 26.25
\end{bmatrix} = 
\begin{bmatrix}
160 \\
51.25
\end{bmatrix}
\]

approximately 160 Juveniles and 51 adults

c) What will happen to the population in the long run?

Solution:

\[
(\lambda - 0.4)(\lambda - 0.75) - (4)(0.5) = 0 \\
\lambda^2 - 1.15\lambda + 0.3 - 2 = 0 \\
\lambda^2 - 1.15\lambda - 1.7 = 0
\]

\[
\lambda = \frac{1.15\pm\sqrt{(1.15)^2-4(-1.7)}}{2} = \frac{1.15\pm\sqrt{1.3225+6.8}}{2} = \frac{1.15\pm\sqrt{8.1225}}{2} = 0.575 \pm 1.425
\]

\(\lambda_1 = 2\) and \(\lambda_2 = -0.85\)
\[
\begin{bmatrix}
0.4 & 4 \\
0.5 & 0.75
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
\mathbf{v}
\end{bmatrix} = 2 \begin{bmatrix}
\mathbf{v}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.4 & 4 \\
0.5 & 0.75
\end{bmatrix}
\begin{bmatrix}
X \\
Y
\end{bmatrix} = 2
\begin{bmatrix}
X \\
Y
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.4X + 4Y \\
0.5X + 0.75Y
\end{bmatrix} = \begin{bmatrix}
2X \\
2Y
\end{bmatrix}
\]

0.4X + 4Y = 2X
0.5X + 0.75Y = 2Y

Y = 0.4X

\[
\begin{bmatrix}
\mathbf{v}
\end{bmatrix} =
\begin{bmatrix}
X \\
Y
\end{bmatrix} =
\begin{bmatrix}
X \\
0.4X
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{v}
\end{bmatrix} =
\begin{bmatrix}
1 \\
0.4
\end{bmatrix}
\]

The dominant eigenvalue is 2 for this discrete time model. Thus, in the long run the populations will grow at a rate of 100% (i.e. they will double every year). And from the corresponding eigen-vector, we see that the long term proportions of juveniles to adults are 1 to 0.4.
1. Let \( \vec{u}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \). The vectors \( \vec{u}_1 \) and \( \vec{u}_2 \) form a basis of \( \mathbb{R}^2 \). What are the coordinates of the point \( \begin{bmatrix} -8 \\ 10 \end{bmatrix} \) with respect to this basis (that is, in the new coordinate system defined by this basis)? (Hint: Write \( \begin{bmatrix} -8 \\ 10 \end{bmatrix} \) as a linear combination of \( \vec{u}_1 \) and \( \vec{u}_2 \).)

Solution:

\[
\begin{bmatrix} -8 \\ 10 \end{bmatrix} = a \begin{bmatrix} 3 \\ -2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\]

(I) \(-8 = 3a + b\)

(II) \(10 = -2a + 4b\)

(III) \(-4 \cdot ( -8 = 3a + b )\)

(IV) \(32 = -12a - 4b\)

(II + IV) \(42 = -14a\)

\(a = -3\)

rearranging equation (I):

\(b = -8 - 3a\)

\(b = -8 - 3(-3)\)

\(b = -8 + 9\)

\(b = 1\)

Thus, the coordinates are: \((-3, 1)\)
2. Let \( \vec{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \). The vectors \( \vec{u}_1 \) and \( \vec{u}_2 \) form a basis of \( \mathbb{R}^2 \), which defines a new coordinate system for \( \mathbb{R}^2 \), which we will call \( R, S \)-coordinates.

(a) What is the matrix \( T \) that converts \( R, S \)-coordinates into standard \( X, Y \)-coordinates? That is, what is the matrix \( T \) in the equation

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = T \begin{bmatrix} R \\ S \end{bmatrix}
\]

\[\text{(2 Points)}\]

Solution:

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = [\vec{u}_1 \quad \vec{u}_2] \begin{bmatrix} R \\ S \end{bmatrix}
\]

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}
\]

\[
T = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}
\]

(b) Expand the equation \( \begin{bmatrix} X \\ Y \end{bmatrix} = T \begin{bmatrix} R \\ S \end{bmatrix} \) into a pair of equations, and solve for \( R \) and \( S \) in terms of \( X \) and \( Y \). Use your result to find the matrix \( T^{-1} \) that converts \( X, Y \) coordinates into \( R, S \) coordinates:

\[
\begin{bmatrix} R \\ S \end{bmatrix} = T^{-1} \begin{bmatrix} X \\ Y \end{bmatrix}
\]

\[\text{(2 Points)}\]

Solution:

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}
\]

(I) \( X = R + 2S \)

(II) \( Y = 3R + 5S \)

Rearranging equation (I):

(III) \( R = X - 2S \)

(II-3I) \(-3X + Y = -6S + 5S \)
S = 3X - Y

Plugging this into equation (III):
R = X - 2(3X - Y)
R = X - 6X + 2Y
R = -5X + 2Y

Thus we have:
R = R = -5X + 2Y
S = S = 3X - Y

Rewriting this as a matrix equation, we get:
\[
\begin{bmatrix}
  R \\
  S 
\end{bmatrix} =
\begin{bmatrix}
  -5 & 2 \\
  3 & -1
\end{bmatrix}
\begin{bmatrix}
  X \\
  Y
\end{bmatrix}
\]

\[
T^{-1} =
\begin{bmatrix}
  -5 & 2 \\
  3 & -1
\end{bmatrix}
\]

(c) The matrix \( M = \begin{bmatrix} -\frac{11}{2} & 2 \\ -15 & \frac{11}{2} \end{bmatrix} \) has \( \vec{u}_1 \) and \( \vec{u}_2 \) as eigenvectors. Verify this using the definition of eigenvectors and eigenvalues. What are their corresponding eigenvalues?

(2 Points)
Solution:

If \( \vec{u}_1 \) and \( \vec{u}_2 \) eigen-vectors of \( M \), then the following must be true, where \( \lambda_1 \) and \( \lambda_2 \) are scalars.

\( M \vec{u}_1 = \lambda_1 \vec{u}_1 \) and \( M \vec{u}_2 = \lambda_2 \vec{u}_2 \)

For \( \vec{u}_1 \):

\[
\begin{bmatrix}
  -\frac{11}{2} & 2 \\
  -15 & \frac{11}{2}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  3
\end{bmatrix} =
\begin{bmatrix}
  0.5 \\
  1.5
\end{bmatrix}
\]

\[
\frac{1}{2} \begin{bmatrix}
  1 \\
  3
\end{bmatrix} = \begin{bmatrix}
  0.5 \\
  1.5
\end{bmatrix}
\]

\( \lambda_1 = \frac{1}{2} \)

For \( \vec{u}_2 \):

\[
\begin{bmatrix}
  -\frac{11}{2} & 2 \\
  -15 & \frac{11}{2}
\end{bmatrix}
\begin{bmatrix}
  2 \\
  5
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  -2.5
\end{bmatrix}
\]

\[
-\frac{1}{2} \begin{bmatrix}
  2 \\
  5
\end{bmatrix} = \begin{bmatrix}
  -1 \\
  -2.5
\end{bmatrix}
\]
\[ \lambda_2 = -\frac{1}{2} \]

(d) Consider the discrete-time dynamical system

\[
\begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} = M \begin{bmatrix} X_n \\ Y_n \end{bmatrix}
\]

Since it’s complicated to understand how this system behaves in \( X, Y \)-coordinates, we want to see what it does to states expressed in \( R, S \)-coordinates instead. We’ll do this in three steps. First, start with the vector \( \begin{bmatrix} R_n \\ S_n \end{bmatrix} \), which represents any arbitrary state, written in \( R, S \)-coordinates. Use your result from part (a) to write \( X_n \) and \( Y_n \) in terms of \( R_n \) and \( S_n \).

(2 Points)
Solution:

\[
\begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} R_n \\ S_n \end{bmatrix} = \begin{bmatrix} R_n + 2S_n \\ 3R_n + 5S_n \end{bmatrix}
\]

(e) Continuing from part (d), take the expressions you found for \( X_n \) and \( Y_n \), and compute

\[ M \begin{bmatrix} X_n \\ Y_n \end{bmatrix} \]

This gives you \( \begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} \) in terms of \( R_n \) and \( S_n \).

(2 Points)
Solution:

\[
\begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} = M \begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \begin{bmatrix} -\frac{11}{15} & 2 \\ -\frac{11}{15} & \frac{2}{15} \end{bmatrix} \begin{bmatrix} R_n + 2S_n \\ 3R_n + 5S_n \end{bmatrix} = \begin{bmatrix} -\frac{11}{15}(R_n + 2S_n) + 2(3R_n + 5S_n) \\ -\frac{11}{15}(R_n + 2S_n) + \frac{11}{15}(3R_n + 5S_n) \end{bmatrix}
\]

\[
= \begin{bmatrix} -\frac{11}{15}R_n - 11S_n + 6R_n + 10S_n \\ -15R_n - 30S_n + \frac{33}{2}R_n + \frac{55}{2}S_n \end{bmatrix} = \begin{bmatrix} \frac{1}{15}R_n - S_n \\ \frac{3}{2}R_n - \frac{5}{2}S_n \end{bmatrix}
\]

\[
\begin{bmatrix} X_{n+1} \\ Y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{15}R_n - S_n \\ \frac{3}{2}R_n - \frac{5}{2}S_n \end{bmatrix}
\]

(f) We want to find \( R_{n+1} \) and \( S_{n+1} \) in terms of \( R_n \) and \( S_n \), but our result from part (e) gave us \( X_{n+1} \) and \( Y_{n+1} \). That is, it gave us the next state in \( X, Y \)-coordinates instead of \( R, S \)-coordinates. Use your result from part (b) to convert the expressions for \( X_{n+1} \) and \( Y_{n+1} \) you found in part (e) into expressions for \( R_{n+1} \) and \( S_{n+1} \), all still in terms of the variables \( R_n \) and \( S_n \) that we started with. What do you notice about the result? What is the matrix \( D \) that makes the following equation work?

\[
\begin{bmatrix} R_{n+1} \\ S_{n+1} \end{bmatrix} = D \begin{bmatrix} R_n \\ S_n \end{bmatrix}
\]

(2 Points)
Solution:

\[
\begin{bmatrix}
R_{n+1} \\
S_{n+1}
\end{bmatrix} = T^{-1} \begin{bmatrix}
X_{n+1} \\
Y_{n+1}
\end{bmatrix} = \begin{bmatrix}
-5 & 2 \\
3 & -1
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} R_n - S_n \\
\frac{3}{2} R_n - \frac{5}{2} S_n
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\frac{5}{2} R_n + 5 S_n + 3 R_n - 5 S_n \\
\frac{3}{2} R_n - 3 S_n - \frac{3}{2} R_n + \frac{3}{2} S_n
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} R_n \\
-\frac{1}{2} S_n
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_{n+1} \\
S_{n+1}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}
\end{bmatrix} \begin{bmatrix}
R_n \\
S_n
\end{bmatrix}
\]

\[D = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}
\end{bmatrix}
\]

(g) Express what you did in parts (d), (e), and (f) as a composition (product) of matrices. The diagram below might be helpful. Then actually multiply these matrices, and verify that it gives you the same matrix \(D\) that you found above.

\[
\begin{align*}
\begin{bmatrix}
X_n \\
Y_n
\end{bmatrix} & \xrightarrow{M} \begin{bmatrix}
X_{n+1} \\
Y_{n+1}
\end{bmatrix} & (X,Y\text{-coordinates}) \\
\begin{bmatrix}
R_n \\
S_n
\end{bmatrix} & \xrightarrow{D} \begin{bmatrix}
R_{n+1} \\
S_{n+1}
\end{bmatrix} & (R,S\text{-coordinates})
\end{align*}
\]

(2 Points)
Solution:

\[
TMT^{-1} = \begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix} \begin{bmatrix}
-\frac{11}{5} & 0 \\
-\frac{11}{5} & 0
\end{bmatrix} \begin{bmatrix}
-5 & 2 \\
3 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 2 \\
3 & 5
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} & 0 \\
\frac{1}{2} & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}
\end{bmatrix}
\]

(h) Using the diagram again, write down an expression for the matrix \(M\) as a composition of other matrices. You don’t need to multiply them out in this case.

(2 Points)
Solution:

\[M = TDT^{-1}\]
3. Compute the absolute values of the following complex numbers:
   (a) $3 + 4i$
   (b) $1 - 2i$
   (c) $0.8 + 0.6i$
   (d) $0.21 - 0.72i$

   (2 Points)

   Solution:
   (a) $\sqrt{3^2 + (4)^2} = 5$
   (b) $\sqrt{(1)^2 + (-2)^2} = \sqrt{5} \approx 2.236$
   (c) $\sqrt{(0.8)^2 + (0.6)^2} = 1$
   (d) $\sqrt{(0.21)^2 + (-0.72)^2} = 0.75$

4. A discrete-time matrix model is defined by the matrix

   $M = \begin{bmatrix}
   -0.21 & -0.84 \\
   1.68 & 1.47
   \end{bmatrix}$

   What kind of behavior do you expect out of this model?

   (2 Points)

   Solution:

   Let's find the eigenvalues of the matrix $M$:

   \[(\lambda + 0.21)(\lambda - 1.47) - (0.84)(-1.68) = 0\]
   \[\lambda^2 - 1.26\lambda - 0.3087 + 1.4112 = 0\]
   \[\lambda^2 - 1.26\lambda + 1.1025 = 0\]

   \[\lambda = \frac{1.26 \pm \sqrt{1.5876 - 4(1.1025)}}{2} = \frac{1.26 \pm \sqrt{-3.47628}}{2}\]

   \[\lambda = 0.63 \pm \frac{1}{2} \sqrt{3.47628}i\]

   \[\lambda \approx 0.63 \pm 1.0912i\]

   \[|\lambda| \approx |0.63 \pm 1.0912i| \approx 1.260\]

   The absolute value of the eigenvalue(s) is greater than 1 and there is an imaginary part. Thus, meaning the populations will grow and oscillate (i.e. outward spiral)
5. In a well-known paper by Crouse, Crowder, and Caswell, the authors set up a discrete-time matrix model for the population of loggerhead sea turtles in the world’s oceans. Their model used seven stages of life (hatchlings, small juveniles, large juveniles, subadults, novice breeders, first-year remigrants, and mature breeders), resulting in a $7 \times 7$ matrix. The approximate eigenvalues of that matrix are as follows:

$$
\begin{align*}
0.372 \\
0.746 + 0.213i \\
0.746 - 0.213i \\
0.265 \\
0.945 \\
-0.088 + 0.120i \\
-0.088 - 0.120i
\end{align*}
$$

What is the dominant eigenvalue? What does this mean about the population of loggerhead sea turtles?

(2 Points)

Solution:

$|0.372| = 0.372$

$|0.746 + 0.213i| = \sqrt{(0.746)^2 + (0.213)^2} \approx 0.7758$

$|0.746 - 0.213i| = \sqrt{(0.746)^2 + (-0.213)^2} \approx 0.7758$

$|0.265| = 0.265$

$|0.945| = 0.945$

$|-0.088 + 0.120i| = \sqrt{(-0.088)^2 + (0.120)^2} \approx 0.1488$

$|-0.088 - 0.120i| = \sqrt{(-0.088)^2 + (-0.120)^2} \approx 0.1488$

$\lambda_{\text{dominant}} = 0.945$

Meaning that in the long run the population of the loggerhead sea turtles will decline at a rate of 5.5%