1. (12 points) In the Serengeti, lions prey upon zebras and gazelles, which in turn compete with each other for resources. Write a system of differential equations to describe this situation, based on the following assumptions:

- In the absence of the other species, the zebra population will grow logistically with a natural per-capita growth rate of 0.03 and a carrying capacity of 600.
- The per-capita birth rate of the gazelle population will be higher when there are few zebras around, and lower when there are many zebras to compete with. Use a decreasing sigmoid function for this, with a maximum of 0.14 (when there are no zebras at all).
- An abundance of gazelles can be harmful to the zebra population. When there are many gazelles, the per-capita death rate of the zebras can increase by as much as 17% per year.
- Each lion preys on zebras at a rate that increases with the abundance of zebras, up to a maximum of 3.4 per year.
- Each lion preys on gazelles at a rate proportional to the gazelle population, with a proportionality constant of 0.03.
- The lion population grows logistically, with a natural per-capita growth rate of 0.07, and a carrying capacity equal to \( \frac{1}{30} \) of the sum of the zebra and gazelle populations.

Note: Although several constants are given in this problem, you may want to introduce others in some places. Feel free to do so.

\[
\begin{align*}
\text{L} & : 0.07L \cdot \left(1 - \frac{L}{600(Z+G)}\right) \\
\text{Z} & : 0.03Z \left(1 - \frac{Z}{600}\right) \\
\text{G} & : 0.14 \cdot \frac{L}{1+Z^m} \\
0.17 \frac{G^k}{1+G^k} & \rightarrow \text{Z} \\
3.4 \frac{Z^m}{1+Z^m} & \rightarrow \text{L}
\end{align*}
\]
\[ L' = 0.07L \cdot \left(1 - \frac{L}{L_0(Z + \theta)}\right) \]
\[ Z' = 0.03Z \cdot \left(1 - \frac{Z}{6000}\right) - 0.17 \cdot \frac{G^k}{1 + G^k} \cdot Z \]
\[ -3.4 \frac{Z_m}{1 + Z_m} \cdot L \]
\[ G' = 0.14 \cdot \frac{1}{1 + Z^n} \cdot G \]
2. (10 points) In a pond ecosystem, green algae produce energy from sunlight via photosynthesis, fish eat the algae and consume some of that energy, and bacteria decompose the dead algae and fish to obtain more of that energy. Set up a discrete-time model to track the energy through such an ecosystem, where $A$ is the amount of energy in the algae, $F$ is the amount of energy in the fish, and $B$ is the amount of energy in the bacteria. Assume the following:

- For each joule of energy in the algae, the algae will produce an additional 0.25 joules each day. (In short, ignoring other factors, the amount of energy in algae will increase by 25% per day.)

- Each day, 12% of the energy in the algae is used up by the algae and leaves the ecosystem, an additional 9% of it ends up in the fish who eat the algae, and an additional 3% of it ends up in the bacteria who eat the dead algae.

- Each day, 36% of the energy in the fish is used up and leaves the ecosystem, and another 7% of it goes to the bacteria.

- Each day, 64% of the energy in the bacteria is used up and leaves the ecosystem.

As usual, a diagram is recommended as a starting point. Write down a discrete-time system of equations for this model. If it is a linear model, also write it in matrix form. If not, explain why it is nonlinear.

![Diagram of energy flow through the ecosystem]

Question 2 continues on the next page...
Equations:
\[
\begin{align*}
A_{t+1} - A_t &= 0.25 A_t - 0.12 A_t - 0.09 A_t - 0.03 A_t \\
F_{t+1} - F_t &= 0.09 A_t - 0.36 F_t - 0.07 F_t \\
B_{t+1} - B_t &= 0.03 A_t + 0.07 F_t - 0.64 B_t
\end{align*}
\]

This is a linear model, so we could also write it in matrix form as
\[
\begin{bmatrix}
A_{t+1} \\
F_{t+1} \\
B_{t+1}
\end{bmatrix} =
\begin{bmatrix}
1.01 & 0 & 0 \\
0.09 & 0.57 & 0 \\
0.03 & 0.07 & 0.36
\end{bmatrix}
\begin{bmatrix}
A_t \\
F_t \\
B_t
\end{bmatrix}
\]
3. (10 points) Some species of insects have populations that follow a delayed version of the logistic differential equation, such as the following:

\[ N'(t) = 0.2N(t - 1.5) \left(1 - \frac{N(t - 1.5)}{100}\right) - 0.1N(t) \]

The table to the right gives many values of the population size, \( N(t) \), at different times \( t \) (in days). Starting from \( t = 0 \) and using a step size of 0.1 days, approximate the value of \( N(0.3) \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N(t) )</th>
<th>( N'(t) )</th>
<th>( N(t + 0.1) \approx N(t + 0.1 \cdot N(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>68</td>
<td>-1.802</td>
<td>68 + 0.1 \cdot (-1.802)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= 67.8198</td>
</tr>
<tr>
<td>0.1</td>
<td>67.8198</td>
<td>-1.832</td>
<td>67.8198 + 0.1 \cdot (-1.832)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= 67.6366</td>
</tr>
<tr>
<td>0.2</td>
<td>67.6366</td>
<td>-1.862</td>
<td>67.6366 + 0.1 \cdot (-1.862)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>= 67.4504</td>
</tr>
<tr>
<td>0.3</td>
<td>67.4504</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>( N(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0</td>
<td>73</td>
</tr>
<tr>
<td>-1.9</td>
<td>68</td>
</tr>
<tr>
<td>-1.8</td>
<td>64</td>
</tr>
<tr>
<td>-1.7</td>
<td>61</td>
</tr>
<tr>
<td>-1.6</td>
<td>58</td>
</tr>
<tr>
<td>-1.5</td>
<td>51</td>
</tr>
<tr>
<td>-1.4</td>
<td>55</td>
</tr>
<tr>
<td>-1.3</td>
<td>57</td>
</tr>
<tr>
<td>-1.2</td>
<td>64</td>
</tr>
<tr>
<td>-1.1</td>
<td>71</td>
</tr>
<tr>
<td>-1.0</td>
<td>66</td>
</tr>
<tr>
<td>-0.9</td>
<td>60</td>
</tr>
<tr>
<td>-0.8</td>
<td>51</td>
</tr>
<tr>
<td>-0.7</td>
<td>53</td>
</tr>
<tr>
<td>-0.6</td>
<td>56</td>
</tr>
<tr>
<td>-0.5</td>
<td>61</td>
</tr>
<tr>
<td>-0.4</td>
<td>67</td>
</tr>
<tr>
<td>-0.3</td>
<td>71</td>
</tr>
<tr>
<td>-0.2</td>
<td>76</td>
</tr>
<tr>
<td>-0.1</td>
<td>72</td>
</tr>
<tr>
<td>0.0</td>
<td>68</td>
</tr>
</tbody>
</table>

\[ \text{Scratch work: Computing } N(t) \]

\[ t=0: \quad N(0) = 0.2 \cdot N(-1.5) \cdot \left(1 - \frac{N(-1.5)}{100}\right) - 0.1N(0) \]
\[ = 0.2 \cdot 51 \cdot \left(1 - \frac{51}{100}\right) - 0.1 \cdot 68 = -1.802 \]

\[ t=0.1: \quad N(0.1) = 0.2 \cdot N(-1.4) \cdot \left(1 - \frac{N(-1.4)}{100}\right) - 0.1N(0.1) \]
\[ = 0.2 \cdot 55 \cdot \left(1 - \frac{55}{100}\right) - 0.1 \cdot 67.8198 = -1.83198 \]

\[ t=0.2: \quad N(0.2) = 0.2 \cdot N(-1.3) \cdot \left(1 - \frac{N(-1.3)}{100}\right) - 0.1N(0.2) \]
\[ = 0.2 \cdot 57 \cdot \left(1 - \frac{57}{100}\right) - 0.1 \cdot 67.6366 = -1.86166 \]
4. (a) (5 points) What two conditions are needed in the state space of a differential equation in order for a limit cycle attractor to exist?

There must be
1. An unstable spiral equilibrium point, and
2. somewhere in the state space around that equilibrium point, other trajectories must be spiralling inward.

(b) (5 points) Suppose you know that a Hopf bifurcation occurs in a system as a parameter is changed. Explain, in terms of eigenvalues, what happens as the Hopf bifurcation occurs. If you want to compute the exact value of the parameter where the Hopf bifurcation occurs, what condition should you solve for?

A Hopf bifurcation occurs when changing a parameter of the system causes a stable spiral equilibrium point to change to an unstable spiral equilibrium point with a limit cycle attractor around it. This means that the eigenvalues of the Jacobian of the system at the equilibrium point change from complex with negative real part to complex with positive real part.

Note that this means the bifurcation occurs at the instant that the real part of these eigenvalues is 0.

So, to solve: find the equilibrium point, the Jacobian at the eq. pt., and then the eigenvalues of this Jacobian, all in terms of the parameter. Then set the real part of the eigenvalues to 0, and solve.
5. (8 points) The following bifurcation diagram illustrates what happens to one variable \((X)\) of the Rössler system of differential equations as a parameter \(a\) is varied from \(a = 2\) to \(a = 10\).

Name two types of bifurcations that can be seen in this diagram. For each type of bifurcation: (1) describe briefly the kind of change in behavior that is associated with it, (2) state how many of these bifurcations occur in the diagram, and (3) specify a value of \(a\) where such a bifurcation occurs.

There is a **Hopf bifurcation**, where the system goes from having a stable equilibrium point to having stable oscillations (limit cycle attractor) at \(a \approx 2.9\).

This is the only Hopf bifurcation.

There are also **period-doubling bifurcations**, where the system goes from having oscillations with period \(n\) to having oscillations with period \(2^n\). There are infinitely many of these in a diagram like this! The first few occur at \(a \approx 3.9, \ a \approx 4.1, \text{ and } a \approx 4.2\).

Another very visible one is at \(a \approx 5.4\).
6. You and your friend are studying a dynamical system whose behavior is chaotic. You each simulate the model for many iterations, but you round the initial conditions to 3 decimal places, whereas your friend uses 4 decimal places.

(a) (5 points) How similar or different should the data from the two simulations be? Explain in terms of specific properties of chaotic systems.

The values from the two simulations should be quite close to each other at first, for the first several time steps. However, they will eventually start to diverge, and in the long run they will end up being completely different from each other.

This is due to sensitive dependence on initial conditions, which says that the distance between the two solutions will grow exponentially over time. Recall that exponential growth is often slow at first, but eventually leads to huge growth.

(b) (5 points) In the long run, what similarities might you see in the simulations? Again, explain this in terms of specific features of chaotic systems.

Chaotic systems have a strange attractor in the state space, which means that even though the two simulations will not look alike in the long run, there will still be patterns of behavior that are the same between the two simulations.

For example, the distribution of states in the state space, the average values, the general shape of the trajectory, etc...
7. The following is a two-stage model of the population of condors in Grand Canyon National Park, in which \( F \) represents fledglings (young birds) and \( A \) represents adults.

\[
\begin{align*}
F_{n+1} &= 0.5F_n + 0.48A_n \\
A_{n+1} &= 0.25F_n + 0.9A_n
\end{align*}
\]

(a) (6 points) In the long run, if the model remains accurate, will this population grow or decline? At what rate?

Long-term behavior: Eigenvalues!

\[
\begin{bmatrix} F_n \\ A_n \end{bmatrix} = \begin{bmatrix} 0.5 & 0.48 \\ 0.25 & 0.9 \end{bmatrix} \begin{bmatrix} F_n \\ A_n \end{bmatrix}
\]

\[
\lambda^2 - (0.5 + 0.9)\lambda + (0.5 \cdot 0.9 - 0.25 \cdot 0.48) = 0
\]

\[
\lambda^2 - 1.4\lambda + 0.33 = 0
\]

\[
\lambda = \frac{1.4 \pm \sqrt{1.4^2 - 4 \cdot 0.33}}{2} = \frac{1.4 \pm \sqrt{0.64}}{2} = \frac{1.4 \pm 0.8}{2}
\]

Eigenvalues: \( \lambda = 0.3, (\lambda = 1.1) \)

The dominant eigenvalue is 1.1, so in the long run, the population will grow at 10% per year.

(b) (6 points) Suppose that the model does remain accurate for a long time, and some time far in the future there are 80 adult condors. How many fledglings will there be at that time?

Need the eigenvector corresponding to the dominant eigenvalue:

\[
\begin{bmatrix} 0.5 & 0.48 \\ 0.25 & 0.9 \end{bmatrix} \begin{bmatrix} F \\ A \end{bmatrix} = 1.1 \begin{bmatrix} F \\ A \end{bmatrix}
\]

\[
\begin{align*}
0.5F + 0.48A &= 1.1F \\
0.25F + 0.9A &= 1.1A
\end{align*}
\]

\[
\begin{align*}
\Rightarrow & \quad -0.6F + 0.98A = 0 \quad \Rightarrow \quad 4A = 5F \\
\Rightarrow & \quad 0.25F - 0.2A = 0 \quad \Rightarrow \quad SF = 9A
\end{align*}
\]

Eigenvector: \( \begin{bmatrix} 4 \\ 5 \end{bmatrix} \)

So the long-term ratio of fledglings to adults is \( \frac{4}{5} \).

So \( \frac{F}{80} = \frac{4}{5} \quad \Rightarrow \quad F = 80 \cdot \frac{4}{5} = 64 \) fledglings.
8. Suppose you know that the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is linear, and

$$f\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ -6 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$  

(a) (3 points) What is $f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

$$f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = f\left(\frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \frac{1}{3} f\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \frac{1}{3} \begin{bmatrix} 9 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$  

(b) (4 points) Using the fact that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

and the properties of linear functions, compute $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = f\left(\frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= f\left(\frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) + f\left(-1 \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= \frac{1}{2} f\left(\begin{bmatrix} 2 \\ 6 \end{bmatrix}\right) + -1 f\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= \frac{1}{2} \begin{bmatrix} 9 \\ -6 \end{bmatrix} + -1 \begin{bmatrix} 8 \\ -6 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix} - \begin{bmatrix} 8 \\ -6 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}.$$  

(c) (3 points) Using your answers to parts (a) and (b), write down the matrix that represents the function $f$.

$$\text{Matrix of } f: \begin{bmatrix} f\left(e_1\right) & f\left(e_2\right) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 7 & -2 \end{bmatrix}.$$
9. The following is a model of the population of wolves ($W$) and elk ($E$) in Yellowstone National Park.

\[
\begin{align*}
W' &= 2EW - 4W - W^2 = f(w,E) \\
E' &= 14E - E^2 - EW = g(w,E)
\end{align*}
\]

(a) (12 points) Find and classify all the equilibrium points for this system.

Step 1: Find the equilibrium points:

\[
\begin{cases}
2EW - 4W - W^2 = 0 \\
14E - E^2 - EW = 0
\end{cases} \Rightarrow \begin{cases}
W(2E-4-W) = 0 \\
W=0 \quad \text{or} \quad 2E-4-W = 0
\end{cases}
\]

\[
\begin{cases}
E(14-E-W) = 0 \\
E=0 \quad \text{or} \quad 14-E-W = 0
\end{cases}
\]

Case 1: $W=0, E=0$: $(0,0)$

Case 2: $W=0, 14-E-W = 0$: $E=14$: $(0,14)$

Case 3: $2E-4-W = 0, E=0$: $W=4$ (*Not in the state space*.

Case 4: $\begin{cases}
2E-4-W = 0 \\
14-E-W = 0
\end{cases}$: $W=14-E$, so $2E-4-(14-E)=0$.

\[
\begin{align*}
3E &= 18 \\
E &= 6 \\
W &= 8
\end{align*}
\]

Case 4: $E=6$: $(8,6)$

Step 2: Jacobian:

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial w} & \frac{\partial f}{\partial E} \\
\frac{\partial g}{\partial w} & \frac{\partial g}{\partial E}
\end{bmatrix} = \begin{bmatrix}
2E-4-2W & 2W \\
-E & 14-2E-W
\end{bmatrix}
\]

Step 3: Classify each eq. point:

$(0,0)$: $J|_{(0,0)} = \begin{bmatrix}
-4 & 0 \\
0 & 14
\end{bmatrix}$

Diagonal matrix, so eigenvalues are $\lambda = -4$ and $\lambda = 14$.

So $(0,0)$ is a saddle point.
Question 9 continued...

\[
J_{(0,14)} = \begin{bmatrix} 2 & 4 \\ -14 & -14 \end{bmatrix}
\]

Lower-triangular matrix, so eigenvalues are \( \lambda = 24 \) and \( \lambda = -14 \).

\( (0,14) \) is a saddle point.

\[
J_{(8,6)} = \begin{bmatrix} -8 & 16 \\ -6 & -6 \end{bmatrix}
\]

Eigenvalues:

\[
\lambda^2 - (-8 + 6) \lambda + (8)(-6) - (16)(-6) = 0
\]
\[
\lambda^2 + 14 \lambda + 144 = 0
\]
\[
\lambda = -7 \pm \sqrt{49 - 144}
\]
\[
\lambda = -7 \pm \sqrt{1 - 4}i
\]
\[
\lambda = -7 \pm i3.87
\]

Eigenvalues are complex with negative real part, so \( (8,6) \) is a stable spiral.

(b) (2 points) What will happen to these species in the long run?

In the long run, all trajectories will spiral into \( (8,6) \).
So eventually the species will coexist at \( W = 8, E = 6 \).
10. Suppose $M$ is a $2 \times 2$ matrix, with the following eigenvectors:

\[
\begin{align*}
    u_1 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{with eigenvalue } \lambda_1 = 3 \\
    u_2 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{with eigenvalue } \lambda_2 = -1
\end{align*}
\]

Assume that, as usual, we use these eigenvectors to define a new coordinate system for $\mathbb{R}^2$ that we call $R, S$-coordinates.

(a) (2 points) Write down the matrix $T$ that converts $R, S$-coordinates to standard $X, Y$-coordinates.

Note that

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = T \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}
\]

means $X = 2R + S$

$Y = 3R + 2S$, so

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = R \begin{bmatrix} 2 \\ 3 \end{bmatrix} + S \begin{bmatrix} 1 \\ 2 \end{bmatrix} = R u_1 + S u_2,
\]

which is the definition of $R, S$-coordinates.

(b) (4 points) Compute the matrix $T^{-1}$.

We know

\[
\begin{bmatrix} X \\ Y \end{bmatrix} = T \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 2R + S \\ 3R + 2S \end{bmatrix}
\]

We want

\[
\begin{bmatrix} R \\ S \end{bmatrix} = T^{-1} \begin{bmatrix} X \\ Y \end{bmatrix}.
\]

Start with

\[
\begin{align*}
    X &= 2R + S \\
    Y &= 3R + 2(X - 2R)
\end{align*}
\]

Solve for $R, S$ in terms of $X, Y$.

\[
\begin{align*}
    S &= X - 2R \\
    Y &= 3R + 2(X - 2R) = 3R + 2X - 4R = 2X - R \\
    R &= 2X - Y
\end{align*}
\]

Question 10 continues on the next page...
Question 10 continued...

\[ S = X - 2R = X - 2(2X - Y) = X - 4X + 2Y = -3X + 2Y \]

So
\[ R = 2X - Y \]
\[ S = -3X + 2Y \]

That is,
\[ \begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \]

So
\[ T^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \]

(c) (3 points) Compute the \( R, S \) coordinates of the point \((3, -5)\), that is, the point that has \( X = 3, Y = -5 \).

Using part (b):
\[ \begin{bmatrix} R \\ S \end{bmatrix} = T^{-1} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -19 \end{bmatrix} \]

That is, \( R = 11, S = -19 \)

(d) (5 points) Compute the matrix \( M \). (For partial credit, at least write down how you could find \( M \), even if you don’t multiply everything out.)

From the diagram (on the next page): \( M = TDT^{-1} \)

where \( D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \)

So
\[ M = TDT^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 9 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 15 & -8 \\ 24 & -13 \end{bmatrix} \]
Some useful formulas, etc:

The characteristic equation of the matrix

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

is \( \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \).

The quadratic formula says that the roots of \( ax^2 + bx + c = 0 \) are

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The diagram for diagonalizing a matrix is

\[
\begin{align*}
T & \quad M \\
\left( \begin{array}{c}
  X_t \\
  Y_t \\
  \vdots
\end{array} \right) & \quad \left( \begin{array}{c}
  X_{t+1} \\
  Y_{t+1} \\
  \vdots
\end{array} \right) \\
T^{-1} & \quad T
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
  R_t \\
  S_t \\
  \vdots
\end{array} \right) & \quad \left( \begin{array}{c}
  \lambda_1 \\
  \lambda_2 \\
  \vdots
\end{array} \right) \\
D & \quad \left( \begin{array}{c}
  R_{t+1} \\
  S_{t+1} \\
  \vdots
\end{array} \right)
\end{align*}
\]