1 Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$

(Material is based on Folland Secs 1.5 and 2.6, but presented a bit differently)

We now complete the construction of Lebesgue measure in $\mathbb{R}^n$ using Carathéodory’s construction, described last time. In $\mathbb{R}^n$, this starts by covering a set of interest with a countable family of rectangles, but not necessarily a finite family as for outer Jordan content.

In fact, one obtains a slightly simpler and cleaner construction of Lebesgue measure if instead of closed rectangles, one uses ‘half-rectangles’. In $\mathbb{R}$, a half-interval or h-interval is a subset of the form $(a, b]$ for some $-\infty \leq a \leq b < \infty$ or $(a, \infty)$ for some $-\infty \leq a < \infty$. In $\mathbb{R}^n$, a half-rectangle or h-rectangle is a set of the form $\prod_{i=1}^n J_i$ for some h-intervals $J_1, \ldots, J_n \subset \mathbb{R}$.

Input for the construction:

- Algebra: all finite unions of h-rectangles.
- Premeasure: for h-rectangles use obvious notion of volume, and extend to disjoint unions of h-rectangles by addition.

In order to extend these data to a measure on a $\sigma$-algebra via Carathéodory’s construction, we must verify that they are indeed an algebra and a premeasure.

First, observe that the family $\mathcal{E}$ of all h-rectangles has the following properties:

- $\emptyset, \mathbb{R}^n \in \mathcal{E}$;
- if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$;
- if $E \in \mathcal{E}$ then $\mathbb{R}^n \setminus E$ is a finite disjoint union of members of $\mathcal{E}$.

(Folland defines an ‘elementary family’ to be any abstract family $\mathcal{E}$ of subsets of some set $X$ having the three properties above: see p23. But I think this term is not very common.)
Proposition 1.1 (Compare Folland Prop 1.7). From the above three properties, it follows that the family of all finite unions of $h$-rectangles is an algebra of sets, and that any member of this algebra may be written as a finite disjoint union of $h$-rectangles.

Moreover:

Proposition 1.2. The obvious notion of volume is a premeasure on the algebra constructed above.

This proposition is not in Folland as stated. When $n = 1$, it is a special case of Folland’s Prop 1.15, and the proof is similar in higher dimensions.

Ideas in the proof. To show that ‘volume’ is subadditive for coverings by finite unions of $h$-rectangles, observe that it agrees with Jordan content for such sets.

To extend this conclusion to coverings by countable unions, use the trick of reducing each $h$-rectangle being covered to a slightly smaller closed rectangle, and enlarging each $h$-rectangle being used in the covering to a slightly bigger open rectangle, and then use that the set being covered is compact.

Definition 1.3. Applying Carathéodory’s construction to the above input gives a complete measure $m^n$ on some $\sigma$-algebra $\mathcal{L}^n$ which contains the Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}^n}$. It is $\sigma$-finite. It is called $n$-dimensional Lebesgue measure\footnote{Sometimes people use this name for the restriction to the Borel sets. In that case it’s not complete.}. If the dimension is clear, we often write $m$ in place of $m^n$.

For $n \geq 2$, we will see a different construction of $m^n$ in terms of $m = m^1$ as a ‘product measure’ later in the course — that construction is the basis of Folland’s Sec 2.6. Once you have learned both, you may work with whichever you prefer.

The construction of Lebesgue measure solves the problem of constructing a very general notion of “$n$-dimensional volume” that we set ourselves at the start of last lecture. The last thing to check is the following:

Theorem 1.4 (Folland Thms 1.21 and 2.42; our proof more like the former). If $E \in \mathcal{L}^n$, $a \in \mathbb{R}^n$, and we define

$$E + a := \{x + a : x \in E\},$$

then $E + a \in \mathcal{L}^n$ and $m^n(E + a) = m^n(E)$. 

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(Other congruences than translations will be handled later.)

It is instructive to compare Lebesgue measure with the older notion of Jordan content. You can find a review of Jordan content on pp71–3 of Folland (reading assignment!). The basic results in that comparison are as follows.

**Lemma 1.5** (Folland Lem 2.43). If \( U \subset \mathbb{R}^n \) is open, then \( U \) is equal to the union of all dyadic cubes that it properly contains. In particular, it is equal to a countable union of cubes with disjoint interiors.

**Corollary 1.6.** Lebesgue measure agrees with inner Jordan content for open sets, and with outer Jordan content for compact sets.

**Remark.** Let \( \mathcal{K}(E) \) and \( \kappa(E) \) be the outer and inner Jordan content of a subset \( E \subset \mathbb{R}^n \), respectively. Using these, define the outer and inner measures of \( E \) to be

\[
\overline{m}(E) = \inf \{ \kappa(U) : E \subset U \text{ open} \} \quad \text{and} \quad \underline{m}(E) = \sup \{ \kappa(K) : E \supset K \text{ compact} \}
\]

That is, the outer measure (resp. inner measure) is a kind of ‘outer-inner’ Jordan content (resp. ‘inner-outer’ Jordan content). Then from the above results one can show that:

A subset \( E \subset \mathbb{R}^n \) is Lebesgue measureable if and only if \( \overline{m}(E) = \underline{m}(E) \), and then this common value is the Lebesgue measure of \( E \).

Folland makes this point only in passing (p73). But if you care only about the problem of measure and integration in \( \mathbb{R}^n \), then you can make Corollary 1.6 the basis of Lebesgue-measure theory\(^2\).

\(^2\)See, for example, Chapter 5 of W. Fleming, *Functions of several variables*, Springer, 1977 for an account written this way. Fleming’s book is freely available online within the UCLA network: use the search function at [link.springer.com](http://link.springer.com).