1 Measurable functions

A $\sigma$-algebra and a measure endow a space with ‘measure-theoretic structure’. Our next task is to analyze functions between such spaces which preserve that structure. Two main reasons to do this:

- It gives us the means to transport calculations or results from one measure space to another.
- It is the first step towards a new theory of integration, in which the functions to be integrated have to relate well to the structure of the underlying measure space.

Let $(X, M)$ and $(Y, N)$ be measurable spaces.

**Definition 1.1.** A function $f : X \to Y$ being $(M, N)$-measurable.

**Lemma 1.2** (Obvious). If $f : X \to Y$ is $(M, N)$-measurable and $g : Y \to Z$ is $(N, O)$-measurable then $g \circ f$ is $(M, O)$-measurable.

**Remark.** At this basic level, it can be helpful to think about measurability as analogous to continuity: recall that a function $f$ is continuous iff $f^{-1}(U)$ is open whenever $U$ is open. But beware of taking this analogy too far: the properties of measurability and openness can behave very differently.

The next proposition is also easy, but extremely important, both for its frequent application and for the general principle which lies behind its proof.

**Proposition 1.3** (Folland Prop 2.1). If $E \subseteq \mathcal{P}(Y)$ and $N$ is the $\sigma$-algebra generated by $E$, then $f : X \to Y$ is $(M, N)$-measurable iff $f^{-1}(E) \in M$ for every $E \in \mathcal{E}$. 

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Corollary 1.4 (Special case of Folland Cor 2.2). If \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is continuous then it is \((\mathcal{B}_{\mathbb{R}^n}, \mathcal{B}_{\mathbb{R}^m})\)-measurable.

Definition 1.5. If \((X, \mathcal{M})\) is a measurable space, then a function \( f : X \rightarrow \mathbb{R} \) (resp. \( f : X \rightarrow \mathbb{C} \)) is \(\mathcal{M}\)-measurable or just measurable if it is \((\mathcal{M}, \mathcal{B}_{\mathbb{R}})\)-measurable (resp. \((\mathcal{M}, \mathcal{B}_{\mathbb{C}})\)-measurable).

In particular, \( f : \mathbb{R} \rightarrow \mathbb{R} \) is Lebesgue (resp. Borel) measurable if it is \((\mathcal{L}, \mathcal{B}_{\mathbb{R}})\)-measurable (resp. \((\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})\)-measurable), and similarly for \( f : \mathbb{R} \rightarrow \mathbb{C} \).

These definitions are also extended in the obvious way to \(\mathbb{R}\)-valued functions, using \( \mathcal{B}_{\mathbb{R}} = \{ E \subset \mathbb{R} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \} \).

Beware: this is a point where the difference between Lebesgue measurable subsets of \( \mathbb{R} \) and Borel subsets is important! In particular, if \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are both Borel measurable, then so is \( g \circ f \), but this is false for Lebesgue measurable functions.

Proposition 1.6 (Folland Prop 2.3). If \((X, \mathcal{M})\) is a measurable space, and \( f : X \rightarrow \mathbb{R} \), then TFAE:

- a. \( f \) is \( \mathcal{M}\)-measurable;
- b. \( f^{-1}((a, \infty)) \in \mathcal{M} \) for all \( a \in \mathbb{R} \);
- c. \( f^{-1}([a, \infty)) \in \mathcal{M} \) for all \( a \in \mathbb{R} \);
- d. \( f^{-1}((-\infty, a)) \in \mathcal{M} \) for all \( a \in \mathbb{R} \);
- e. \( f^{-1}((-\infty, a]) \in \mathcal{M} \) for all \( a \in \mathbb{R} \);

If \( f : X \rightarrow \mathbb{R}, \mathcal{M} \) is a \( \sigma \)-algebra of subsets of \( X \), and \( E \in \mathcal{M} \), then \( f \) is measurable on \( E \) if \( f^{-1}(B) \cap E \in \mathcal{M} \) for all \( B \in \mathcal{B}_{\mathbb{R}} \).