245A: Lecture 9

For the whole of this lecture, fix a measure space \((X, \mathcal{M}, \mu)\).

1 Integration of nonnegative functions, I

(From Folland Sec 2.2)

Let

\[ \mathcal{L} = \text{the space of all measurable functions from } X \text{ to } [0, \infty]. \]

**Definition 1.1.** If \( \varphi \) is a non-negative simple function (so \( \varphi \in \mathcal{L} \)) with standard representation \( \sum_{j=1}^{n} a_j \chi_{E_j} \), then its integral is

\[ \int \varphi \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j) \]

(\text{where, as always, } 0 \cdot \infty = 0). \]

In case \( X = [a, b] \), any nonnegative step function (i.e., a function which dissects \( [a, b] \) into finitely many subintervals and is constant on each) is certainly simple, and then its integral as above clearly matches its Riemann integral. But the first great advantage of our new integral starts to show itself: any simple function built out of measurable sets can now be integrated, even if those measurable sets are much more complicated than intervals.

This value may equal \( +\infty \). Other notations (chosen depending on context):

\[ \int \varphi \, d\mu = \int \varphi = \int \varphi(x) \, d\mu(x). \]
If $A \in \mathcal{M}$, then $\varphi \chi_A$ still simple, and we define
\[
\int_A \varphi = \int \varphi \chi_A.
\]
In particular, $\int = \int_X$.

**Proposition 1.2** (Folland Prop 2.13). Let $\varphi$ and $\psi$ be simple functions in $L^+$.

a. If $c \geq 0$ then $\int c \varphi = c \int \varphi$.

b. $\int (\varphi + \psi) = \int \varphi + \int \psi$.

c. If $\varphi \leq \psi$ then $\int \varphi \leq \int \psi$.

d. The map $A \mapsto \int_A \varphi$ is a measure on $\mathcal{M}$.

Parts a. and b. are called the **linearity** of the integral, and part c. is its **monotonicity**.

For the proof: all parts are easy, except that in parts b. and c. we have to carefully compare the standard representations of $\varphi$ and $\psi$. The idea is to take the sets $E_j$ and $F_k$ appearing in those two standard representations, and instead re-write all the integrals of interest in terms of the intersections $E_j \cap F_k$.

**Definition 1.3.** If $f \in L^+$, then
\[
\int f \, d\mu = \sup \left\{ \int \varphi \, d\mu : 0 \leq \varphi \leq f, \varphi \text{ simple} \right\}.
\]
(Note: if $f$ itself is simple, then this agrees with the previous definition, part monotonicity.)

**Lemma 1.4** (Obvious). If $f \leq g$ in $L^+$ then $\int f \leq \int g$, and if $c \geq 0$ then $\int cf = c \int f$.

We will see that integration is still linear on $L^+$, but before that we need to prove our first major convergence theorem.

**Theorem 1.5** (The monotone convergence theorem; Folland Thm 2.14). If $\{f_n\}$ is a sequence in $L^+$ such that $f_n \leq f_{n+1}$ for all $n$, and if $f = \lim_{n \to \infty} f_n$ ($= \sup_n f_n$), then $\int f = \lim_{n \to \infty} \int f_n$.  

2
Remarks on the proof. The tricky part here is to prove the inequality $\leq$. To do this, let $\varphi \leq f$ be a non-negative simple function, and show that for any $\alpha < 1$ we have $\alpha \int \varphi \leq \lim \int f_n$. That last conclusion can be obtained from the continuity of $\mu$ under increasing unions.

First reason why the monotone convergence theorem is important: the definition of $\int f \, d\mu$ for $f \in L^+$ involves a supremum over a potentially enormous family of simple functions, but in fact we can obtain it as $\lim \int \varphi_n$ from any sequence of simple functions which converge pointwise to $f$ from below.