Problem 1

In the standard SI units, the electric constant \( \varepsilon_0 = 8.854 \times 10^{-12} \text{ s}^2 \cdot \text{C}^2 \cdot \text{m}^{-3} \cdot \text{kg}^{-1} \). In the rationalized system of SI units, one sets \( \varepsilon_0 = 1 \). Express the proton electric charge \( e = 1.602 \times 10^{-19} \text{ C} \) in this system in terms of units (kg, m, s). In the natural units, one sets \( \hbar = c = 1 \). What is proton electric charge \( e \) in the natural units?

In SI units, Coulomb is an independent unit, while in rationalized SI units, it is NOT.

In standard SI units:
\[
\varepsilon_0 = 8.854 \times 10^{-12} \text{ s}^2 \cdot \text{C}^2 \cdot \text{m}^{-3} \cdot \text{kg}^{-1}
\]

In rationalized SI units:
\[
\varepsilon_0 = 1
\]

Consequently,
\[
1 = 8.854 \times 10^{-12} \text{ s}^2 \cdot \text{C}^2 \cdot \text{m}^{-3} \cdot \text{kg}^{-1}
\]

\[
\Rightarrow 1 \text{C}^2 = \frac{1}{8.854 \times 10^{-12}} \text{ s}^2 \cdot \text{m}^3 \cdot \text{kg}
\]

\[
\Rightarrow 1 \text{C} = \frac{1}{\sqrt{8.854 \times 10^{-12}}} \text{ kg}^{\frac{1}{2}} \cdot \text{s}^{-1} \cdot \text{m}^{\frac{1}{2}} \approx 3.36 \times 10^5 \text{ kg}^{\frac{1}{2}} \cdot \text{s}^{-1} \cdot \text{m}^{\frac{1}{2}}
\]

Proton charge in SI units, \( e = 1.602 \times 10^{-19} \text{ C} \)

In rationalized SI units:
\[
e = 1.602 \times 10^{-19} \times 3.36 \times 10^5 \text{ kg}^{\frac{1}{2}} \cdot \text{s}^{-1} \cdot \text{m}^{\frac{1}{2}} = 5.38 \times 10^{-14} \text{ kg}^{\frac{1}{2}} \cdot \text{s}^{-1} \cdot \text{m}^{\frac{1}{2}}
\]

In natural units \( \hbar = c = 1 \):
\[
\alpha = \frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{\hbar c} = \frac{1}{137}
\]

\[
\Rightarrow e = \sqrt{\frac{4\pi \varepsilon_0}{137}} = 9.01 \times 10^{-7} \text{ C} \cdot \text{m}^{\frac{1}{2}} \cdot \text{kg}^{-\frac{1}{2}}
\]

“Rationalized” system: in natural units \( \hbar = c = 1 \), also \( \varepsilon_0 = 1 \):

\[
\alpha = \frac{e^2}{4\pi} \Rightarrow e = \frac{4\pi}{137} \Rightarrow e \approx 0.3028
\]
Problem 2

In the standard SI units, the Boltzmann constant $k_B = 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 8.62 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$. However, in the natural units, one sets $k_B = 1$. Why can one set $k_B = 1$? What does it mean? In such a unit system, express the temperature 100 K in terms of eV.

<table>
<thead>
<tr>
<th>In the same way, in standard SI system Kelvin and eV are independent units. Nevertheless, if we put $k_B = 1$ they are not independent anymore, the temperature is measured in the units of energy.</th>
<th>Such a system of units is used, for instance, in Cosmology where it is convenient. In different epochs with different temperatures different types of fields dominate.</th>
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<tbody>
<tr>
<td>$1 = k_B = 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 8.62 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$</td>
<td>$100 = k_B = 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 8.62 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$</td>
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<tr>
<td>$\Rightarrow K = 8.62 \times 10^{-5} \text{ eV}$</td>
<td>$\Rightarrow 100K = 8.62 \times 10^{-3} \text{ eV}$</td>
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</table>
Problem 3

If the kinetic energy of the $\alpha$ particles is 4 MeV, what is their velocity $v$ if you assume them to be nonrelativistic? How large an error do you make in neglecting special relativity in the calculation of $v$?

Nonrelativistically,

$$T = \frac{1}{2}mv^2$$
$$v = \sqrt{\frac{2T}{m}}$$
$$v \approx 1389389\text{m/s}$$

Relativistically,

$$T = (\gamma - 1)mc^2$$
$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1 + \frac{T}{mc^2}$$
$$v = c \sqrt{1 - \frac{1}{(1 + \frac{T}{mc^2})^2}}$$
$$v \approx 13887214\text{m/s}$$

The error between these two values may be expressed as

$$\text{error} = \frac{v_R - v_{NR}}{v_R}$$
$$\text{error} \approx .08\%$$
Problem 4

An electron of momentum $0.511 \text{ MeV}/c$ is observed in the laboratory. What are its $\beta = v/c$, $\gamma = (1 - \beta^2)^{-1/2}$, kinetic energy, and total energy?

| Rest mass of the electron, $m = 0.511 \text{MeV}/c^2$ ⇒ $m \cdot c^2 = 0.511 \text{MeV}$ |
|-----------------|-----------------|
| Momentum of the electron, $p = 0.511 \text{MeV}/c$ ⇒ $p \cdot c = 0.511 \text{MeV}$ |
| As the electron is fairly relativistic, energy $E = \sqrt{p^2 \cdot c^2 + m^2 \cdot c^4} = \sqrt{(0.511)^2 + (0.511)^2 \text{MeV}} = \sqrt{2} \cdot 0.511 \text{MeV} = 0.722 \text{MeV} = \sqrt{2}m \cdot c^2$ |
| Also $E = \gamma \cdot m \cdot c^2$ ⇒ $\gamma = \sqrt{2}$ |
| $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ ⇒ $\beta = \sqrt{1 - \frac{1}{\gamma}}$ ⇒ $\beta = \frac{1}{\sqrt{2}} = 0.707$ |
| Kinetic energy $T = (\gamma - 1) \cdot m \cdot c^2 = (\sqrt{2} - 1) \cdot 0.511 \text{MeV} = 0.211 \text{MeV}$ |
Problem 5

In Rutherford scattering experiment as shown in Fig. 1, an $\alpha$-particle with the mass $m$ and the initial velocity $v$ is scattering from an atomic nucleus of charge $+Ze$ and mass $M$. Please answer the following questions.

(a) Assuming the $\alpha$-particle is deflected from its original direction by an angle $\theta$ as shown in the figure. Please derive the change in its momentum: $\Delta p$.

Before and after the scattering, by conservation of energy, the $\alpha$-particle will have momentum $mv$. These two momenta may be thought of as the legs of an isosceles triangle with an angle $\theta$ between them. Therefore, the change in the momentum, $\Delta p$, is equal to the other side of this triangle: $2mv \sin \frac{\theta}{2}$.

(b) Derive a formula for the closest distance $r_{\text{min}}$ between the $\alpha$-particle and the nucleus.

Assuming that the impact parameter, $b$, is zero, here I set the initial kinetic energy equal to the potential energy at the distance $r_{\text{min}}$.

\[
\frac{1}{2}mv^2 = \frac{2Ze^2}{4\pi \varepsilon_0 r_{\text{min}}} \\
\Rightarrow r_{\text{min}} = \frac{Ze^2}{\pi \varepsilon_0 m v^2}
\]

Using the equation derived in lecture b
\[
\frac{A}{2\alpha_{\text{em}}} \cot \frac{\theta}{2},
\]
for scattering of $1^\circ$,

\[
b \approx \frac{(79)(197)(114.6)}{(7.7 \times 10^6)(137)} \text{ nm} \\
b \approx 1690 \text{ fm}
\]

For the case of $30^\circ$,

\[
b \approx \frac{(79)(197)(3.732)}{(7.7 \times 10^6)(137)} \text{ nm} \\
b \approx 55.2 \text{ fm}
\]

The ratio of probabilities may be calculated by taking the ratio of the cross sections.

\[
P = \frac{\int_0^\frac{\pi}{2} d\theta \sin \theta}{\int_0^\frac{\pi}{2} d\theta \sin \theta} \approx 943
\]