1. (a) Define: \( \lim_{n \to \infty} s_n = s \). (b) Prove that if \( s_n \neq 0 \) for all \( n \) and \( \lim_{n \to \infty} s_n = s \neq 0 \), then \( \lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{s} \). NOTE: You may assume that there exists \( m > 0 \) such that \( |s_n| \geq m \) for all \( n \).

(5 points) (a) Given \( \varepsilon > 0 \) there exists \( N \) such that \( n > N \) implies \( |s_n - s| < \varepsilon \).

(15 points) (b) Given \( \varepsilon > 0 \) there exists \( N \) such that \( n > N \) implies \( |s_n - s| < m \varepsilon \). Then \( |s_n| \geq m > 0 \) implies

\[
\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s - s_n|}{|s_n| |s|} < \frac{m |s| \varepsilon}{m |s|} = \varepsilon
\]

so \( \lim_{n \to \infty} \frac{1}{s_n} = \frac{1}{s} \).
2. (a) Let \( S \subseteq \mathbb{R} \) be a bounded subset. Define \( \inf(S) \), the \textit{infimum} of \( S \).

(b) Let \( A, B \subseteq \mathbb{R} \) be bounded subsets of \( \mathbb{R} \) then, defining \( A + B = \{a + b : a \in A, b \in B\} \), prove that \( \inf(A + B) \geq \inf(A) + \inf(B) \).

(c) Prove that \( \inf(A + B) \leq \inf(A) + \inf(B) \).

(5 points) (a) \( \inf(S) \leq t \) for all \( t \in S \) and if \( t \leq s \) for all \( s \in S \) then \( t \leq \inf(S) \).

(5 points) (b) \( \inf(A) \leq a \) for all \( a \in A \) and \( \inf(B) \leq b \) for all \( b \in B \), so \( \inf(A) + \inf(B) \) is a lower bound for \( A + B \) and thus \( \inf(A) + \inf(B) \leq \inf(A + B) \).

(10 points) (c) \( \inf(A + B) \leq a + b \) so \( \inf(A + B) - b \leq a \) for all \( a \in A \), so \( \inf(A + B) - b \leq \inf(A) \), then \( \inf(A + B) - \inf(A) \leq b \) for all \( b \in B \). So

\[
\inf(A + B) - \inf(A) \leq \inf(B) \\
\inf(A + B) \leq \inf(A) + \inf(B)
\]
3. (a) Define: \( \lim_{n \to \infty} s_n = -\infty \). (b) Define: \( \lim \sup s_n \). (c) Prove that \( \lim \sup s_n = -\infty \) implies \( \lim_{n \to \infty} s_n = -\infty \).

(5 points) (a) Given \( M < 0 \) there exists \( N \) such that
\[
n < N \text{ implies } s_n < M.
\]

(5 points) (b) \( \lim \sup s_n = \lim_{N \to \infty} \nu_N \) where
\[
\nu_N = \sup \{ s_n : n > N \}
\]

(10 points) (c) Given \( M < 0 \) there exists \( N_0 \in \mathbb{N} \) such that \( N > N_0 \) implies \( \nu_N < M \) so \( s_n < M \) for all \( n > N_0 \) and thus \( \lim_{n \to \infty} s_n = -\infty \).
4. (a) Define: \((s_n)\) is a decreasing sequence. (b) Prove that the sequence defined by \(s_1 = 1\) and
\[
 s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2
\]
converges.

(5 points) (a) \(s_{n+1} \leq s_n\) for all \(n\)

(15 points) (b) \(s_2 = \frac{1}{2}\) so \(P_1: 0 \leq s_2 \leq 1\).
Induction hypothesis:
\(P_n: 0 \leq s_{n+1} \leq s_n \leq 1\)

\(P_{n+1}: 0 \leq s_{n+2} = \left(\frac{n+1}{n+2}\right) s_{n+1}^2 \leq s_n \leq 1\)

Since \(s_{n+1} \leq 1\). So \((s_n)\) is a decreasing sequence bounded below by 0 and therefore it converges.
5. (a) Let \( (s_n) \) be a sequence such that \( s_n > 0 \) for all \( n \). Prove that if \( (s_n) \) is unbounded, then \( S_N = \{ s_n : n > N \} \) is unbounded for all \( N \in \mathbb{N} \). (b) Let \( (a_n), (b_n) \) be sequences such that \( a_n > 0, b_n > 0 \) for all \( n \). Prove that if \( \lim_{n \to \infty} a_n/b_n = 1 \) and \( (a_n) \) is a bounded sequence, then \( (b_n) \) is also a bounded sequence.

(7 points) (a) If \( S_N \) is bounded then there exists \( M > 0 \) such that \( s_n < M \) for all \( n > N \). But then \( s_n \leq \max \{ s_N, \ldots, s_N, 1/M \} \) for all \( n \geq N \) so \( (s_n) \) would be bounded.

(13 points) (b) Since \( (a_n) \) is bounded there exists \( M > 0 \) such that \( a_n < M \) for all \( n \).

Since \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \) then given \( \varepsilon > 0 \) (assume \( \varepsilon < 1 \)) then \( \left| \frac{a_n}{b_n} - 1 \right| < \varepsilon \) for \( n > N_{\varepsilon} \).

\(- \varepsilon < \frac{a_n}{b_n} - 1 < \varepsilon \) and \( 1 - \varepsilon < \frac{a_n}{b_n} \) so

\( b_n < \frac{a_n}{1 - \varepsilon} < \frac{M}{1 - \varepsilon} \) and \( b_n < \max \{ b_1, \ldots, b_N, \frac{M}{1 - \varepsilon} \} \).