Problem 1

For any given spatial distribution $\rho (\vec{r})$, one can define the form factor of a target in terms of its Fourier transform $F (q^2)$ in momentum transfer $q$, as given

$$F (q^2) = \int d^3\vec{r} \rho (\vec{r}) e^{i\vec{q}\cdot\vec{r}},$$

where $\rho (\vec{r})$ is normalized to unity: $\int d^3\vec{r} \rho (\vec{r}) = 1$. Now assuming that the target is a hard sphere, i.e.,

$$\rho (\vec{r}) = \begin{cases} \rho_0, & r \leq R, \\ 0, & r > R. \end{cases}$$

Please

(a) determine $\rho_0$ in terms of $R$.

$$\int d^3\vec{r} \rho (\vec{r}) = 1 = \int_0^R d^3\vec{R} \rho_0 = \frac{4}{3} \pi R^3 \rho_0 \Rightarrow \rho_0 = \frac{3}{4\pi R^3}$$

$\Rightarrow \rho (r) = \begin{cases} \frac{3}{4\pi R^3}, & r \leq R, \\ 0, & r > R. \end{cases}$

(b) derive the form factor $F (q^2)$.

$$F (q^2) = \int d^3\vec{r} \rho (\vec{r}) e^{i\vec{q}\cdot\vec{r}} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^R drr^2 \sin\theta \rho_0 e^{\pm i|q|r \cos\theta} = 2\pi \int_0^R drr^2 \rho_0 \frac{h}{|q|r} \left[ e^{\pm i|q|r} - e^{-\pm i|q|r} \right]$$

$$= 2\pi \int_0^R drr\rho_0 \frac{h}{|q|r} \left( 2i \sin \frac{|q|r}{R} \right) = \frac{4\pi h \rho_0}{|q|} \int_0^R drr \sin \frac{|q|r}{R} = \frac{4\pi h \rho_0}{|q|} \left[ \frac{h^2}{|q|^2} \sin \frac{|q|R}{R} - \frac{hR}{|q|} \cos \frac{|q|R}{R} \right] = \frac{3}{\alpha} \left[ \sin \alpha - \alpha \cos \alpha \right]$$

where $\alpha = \frac{|q|R}{h}$. 


Problem 2

The semi-empirical mass formula provides a good approximation for the mass of a nucleus. For a nucleus with mass number $A$ and atomic number $Z$, the mass of the nucleus is given by

$$M(A, Z) = (A - Z) m_n + Z m_p - a_v A + a_v A^{2/3} + a_c \frac{Z^2}{A^{1/3}} + a_s \frac{(A - 2Z)^2}{4A} + \delta \frac{A^{1/2}}{A},$$

where $a_v = 15.67 \text{ MeV}/c^2$, $a_s = 17.23 \text{ MeV}/c^2$, $a_c = 0.714 \text{ MeV}/c^2$, $a_a = 93.15 \text{ MeV}/c^2$, and

$$\delta = \begin{cases} 
-11.2 \text{ MeV}/c^2 & \text{for even } Z \text{ and } N \text{ (even-even nuclei)} \\
0 \text{ MeV}/c^2 & \text{for odd } A \\
+11.2 \text{ MeV}/c^2 & \text{for odd } Z \text{ and } N \text{ (odd-odd nuclei)}
\end{cases}$$

Answer the following questions:

(a) Please show explicitly that, for fixed $A$, $M(A, Z)$ has a minimum value. What does this imply about the stability of the nucleus?

Taking the derivative with respect to $Z$ while keeping $A$ constant and setting this equal to zero will give us the minimum value.

$$\frac{\partial M}{\partial Z} = 0 = -m_n + m_p + a_c \frac{2Z}{A^{1/3}} - 4a_s \frac{(A - 2Z)}{4A}$$

$$Z = \frac{A m_n - m_p + a_s}{2} \frac{a_c A^{2/3} + a_a}{a_a}$$

To check that this is indeed a minimum, we check the sign of the second derivative.

$$\frac{\partial^2 M}{\partial Z^2} = \frac{2 a_c}{A^{1/3}} + \frac{2 a_a}{A} > 0$$

This implies that there is a number of protons for which the nucleus is the most stable for a given value of $A$.

(b) What is the stabllest nucleus with $A = 16$ and $A = 208$, respectively?

$Z_{\min}(A = 16) \approx 7.6$, corresponding to $^{16}$O.

$Z_{\min}(A = 208) \approx 82.2$, corresponding to $^{208}$Pb.

(c) Calculate the binding energy per nucleon for nuclei $^{208}$Pb and $^{197}$Au.

The binding energy per nucleon is

$$\frac{B(A, Z)}{A} = a_v - \frac{a_s}{A^{1/3}} - a_c \frac{Z^2}{A^{4/3}} - a_a \frac{(A - 2Z)^2}{4A^2} - \frac{\delta}{A^{3/2}}$$

For lead, the binding energy per nucleon is 7.83 MeV.

For gold, the binding energy per nucleon is 7.91 MeV.
Problem 3

In the Fermi gas model, please
(a) derive the average kinetic energy per nucleon \( \langle E_{\text{kin}} \rangle \), first write your result in terms of Fermi momentum \( p_F \), and then re-express your result in terms of Fermi energy \( E_F \).

\[
\langle E_{\text{kin}} \rangle = \int \frac{p^2}{2m} dn = \int_0^{p_F} \frac{p^2}{2m} p^2 dp = \frac{1}{2m} \left( \frac{1}{3} p_F^5 - \frac{1}{5} p_F^3 \right) = \frac{3 p_F^5}{10m} = \frac{3 E_F}{5}
\]

(b) show that the Fermi pressure is given by \( p = \frac{2}{5} \rho N E_F \), where \( \rho N = A/V \) is the nucleon density. Note: pressure is given by \( p = -\frac{\partial U}{\partial V} \), where \( U = A \langle E_{\text{kin}} \rangle \) the average kinetic energy per nucleon.

The internal energy may be expressed as

\[ U = A \langle E_{\text{kin}} \rangle. \]

Now \( p_F \) needs to be expressed in terms of \( A \) and \( V \).

\[
A = \frac{8\pi}{\hbar^3} V \int_0^{p_F} p^2 dp = \frac{8\pi V p_F^3}{3\hbar^3}
\]

\[ p_F = \left( \frac{3A h^3}{8\pi V} \right)^{1/3} \]

Therefore,

\[
U = \frac{3A}{10m} \left( \frac{3A h^3}{8\pi V} \right)^{2/3}
\]

\[
P = -\frac{\partial U}{\partial V} = 2 \frac{3A}{10mV} \left( \frac{3A h^3}{8\pi V} \right)^{2/3}
\]

\[
= \frac{A}{5mV} p_F^2 = \frac{2 A}{5 V} E_F = \frac{2}{5} \rho N E_F
\]
Problem 4

Density related questions

(a) Make the approximation that the $^{12}$C nucleus is a uniform sphere of radius $1.2A^{1/3}$ fm. Calculate the weight of a cubic centimeter of material of the same density in pounds.

$$\frac{1.993 \times 10^{-26}\text{kg}}{\frac{4}{3}\pi (1.2 \times 10^{-15} \times 12^{1/3}\text{m})^3} \cdot \frac{1\text{lb}}{.454\text{kg}} \cdot \frac{1\text{m}^3}{(100\text{cm})^3} \approx 5.05 \times 10^{11}\text{lbs./cm}^3$$

(b) The density of nuclear matter is equal to 0.17 nucleons/fm$^3$. The mean density of the neutron star is equal to $10^{18}$ kg/m$^3$. What is the ratio of the density of a neutron star to the density of nuclear matter?

Given that neutron stars are made of neutrons, the density is

$$\frac{10^{18}\text{kg}}{\text{m}^3} \cdot \frac{1\text{nucleon}}{1.675 \times 10^{-27}\text{kg}} \cdot \frac{1\text{m}^3}{(10^{15}\text{fm})^3} \approx .597\text{nucleons/fm}^3.$$ Therefore, the ratio is approximately 3.51.
Problem 5

Show explicitly that

\[ e^{i\vec{q} \cdot \vec{x}/\hbar} = -\frac{\hbar^2}{|\vec{q}|^2} \nabla^2 e^{i\vec{q} \cdot \vec{x}/\hbar} \]

Hint: \[ \vec{q} \cdot \vec{x} = \sum_{i=1}^{3} q_i x_i. \]

\[
\begin{align*}
-\frac{\hbar^2}{|\vec{q}|^2} \nabla^2 e^{i\vec{q} \cdot \vec{x}} &= -\frac{\hbar^2}{|\vec{q}|^2} \left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right] e^{i(q_1x_1 + q_2x_2 + q_3x_3)} \\
&= -\frac{\hbar^2}{|\vec{q}|^2} \left[ \left( \frac{iq_1}{\hbar} \right)^2 + \left( \frac{iq_2}{\hbar} \right)^2 + \left( \frac{iq_3}{\hbar} \right)^2 \right] e^{i(q_1x_1 + q_2x_2 + q_3x_3)} \\
&= \frac{1}{|\vec{q}|^2} \left( q_1^2 + q_2^2 + q_3^2 \right) e^{i\vec{q} \cdot \vec{x}} \\
&= e^{i\vec{q} \cdot \vec{x}}
\end{align*}
\]