Let $X$ be a nonempty set. A partition of $X$ is a collection $\mathcal{P}$ of subsets of $X$ such that every $x \in X$ lies in exactly one member of $\mathcal{P}$. The members of $\mathcal{P}$ are called its parts.

An algebra of subsets of $X$ is a collection $\mathcal{A}$ of subsets of $X$ which contains $\emptyset$ and is closed under complements and finite unions.

If $\mathcal{P}$ is a finite partition of $X$, then algebra generated by $\mathcal{P}$ is the set of all unions of members of $\mathcal{P}$. (Easy exercise: this is, in fact, an algebra of sets.)

If $\mathcal{E}$ is a collection of subsets of $X$, then we sometimes refer to the members of $\mathcal{E}$ as $\mathcal{E}$-sets.

In this supplementary note, we use partitions to describe when certain collections of finite unions form an algebra. This provides an alternative route to Folland’s Proposition 1.7 and its applications in his Sections 1.5 (for finite unions of h-intervals) and 2.5 (for finite unions of measurable rectangles).

The next piece of terminology is not standard, but is convenient in the sequel.

**Definition 1.** Let $X$ be a set, $E \subset X$, and $\mathcal{P}$ a partition of $X$. Then $E$ is **decomposed by** $\mathcal{P}$ if $E$ is a union of some of the parts of $\mathcal{P}$.

Here is our main result about families of sets generating algebras.

**Proposition 2.** Let $\mathcal{E}$ be a family of subsets of $X$ with the following property:

(*) If $E_1, \ldots, E_n \in \mathcal{E}$, then there is a finite partition $\mathcal{P} \subset \mathcal{E}$ which decomposes every $E_i$.

Let $\mathcal{A}$ be the set of all finite unions of $\mathcal{E}$-sets (including $\emptyset$, which is the union of the empty family of subsets of $X$). Then (a) every member of $\mathcal{A}$ can be written as a finite disjoint union of $\mathcal{E}$-sets, and (b) $\mathcal{A}$ is an algebra of sets.
Proof. Part (a). Given $E_1, \ldots, E_n \in \mathcal{E}$, let $\mathcal{P} \subset \mathcal{E}$ be a finite partition which decomposes every $E_i$. This means that each $E_i$ can be written as

$$E_i = \bigcup_{P \in \mathcal{P}_i} P$$

for some subcollections $\mathcal{P}_i \subset \mathcal{P}$. So we have

$$E_1 \cup \cdots \cup E_n = \bigcup_{P \in \mathcal{P} \cup \cdots \cup \mathcal{P}_n} P.$$  

(1)

The right-hand side is a finite disjoint union of $\mathcal{E}$-sets.

Part (b). The family $\mathcal{A}$ contains $\emptyset$ and is closed under finite unions by definition. It remains to show that it is closed under complements. This follows by re-using the formula (1): taking complements in that formula gives

$$(E_1 \cup \cdots \cup E_n)^c = \bigcup_{P \in \mathcal{P} \setminus (\mathcal{P} \cup \cdots \cup \mathcal{P}_n)} P,$$

which is manifestly another finite union of $\mathcal{E}$-sets, and thus a member of $\mathcal{A}$. □

Examples 3.

1. If $\mathcal{E}$ is itself an algebra of subsets of $X$ then it satisfies $(\ast)$. Indeed, if $E_1, \ldots, E_n \in \mathcal{E}$, then also $E_1^c, \ldots, E_n^c \in \mathcal{E}$, and any union or intersection constructed from these sets is in $\mathcal{E}$. Now write $E_{i,1} := E_i$ and $E_{i,0} := E_i^c$ for each $i = 1, \ldots, n$. All of the intersections

$$P_{\omega_1, \ldots, \omega_n} := E_{1, \omega_1} \cap E_{2, \omega_2} \cap \cdots \cap E_{n, \omega_n} \quad \text{for} \quad (\omega_1, \ldots, \omega_n) \in \{0, 1\}^n$$

lie in $\mathcal{E}$, and together these sets constitute a partition of $X$ which decomposes every $E_i$: specifically, we have

$$E_i = \bigcup_{(\omega_1, \ldots, \omega_{i-1}, 1, \omega_{i+1}, \ldots, \omega_n) \in \{0, 1\}} P_{\omega_1, \ldots, \omega_{i-1}, 1, \omega_{i+1}, \ldots, \omega_n}$$

for each $i$.

2. Let $X = \mathbb{R}$ and let $\mathcal{E}$ be the collection of all $h$-intervals. If $E_1, \ldots, E_n \in \mathcal{E}$, then they are defined by finitely many end-points. Ignoring $\pm \infty$, let us
enumerate all of these end-points as $a_1 < a_2 < \cdots < a_N$. Then each $E_i$ is decomposed by the partition

$$R = (-\infty, a_1] \cup (a_1, a_2] \cup (a_{N-1}, a_N] \cup (a_N, \infty),$$

which consists of $\mathcal{E}$-sets.

3. (A variant on the previous example.) Let $X = \mathbb{R}$ and let $\mathcal{E}$ be the collection of all intervals (open or closed on either end). If $E_1, \ldots, E_n \in \mathcal{E}$, then they are defined by finitely many end-points. Ignoring $\pm \infty$, let us enumerate all of these end-points as $a_1 < a_2 < \cdots < a_N$. Then each $E_i$ is decomposed by the partition

$$R = (-\infty, a_1) \cup \{a_1\} \cup (a_1, a_2) \cup \{a_2\} \cup \cdots \cup \{a_N\} \cup (a_N, \infty),$$

which consists of $\mathcal{E}$-sets.

4. Let $X$ and $Y$ be two nonempty sets, let $\mathcal{E}$ and $\mathcal{F}$ be families of subsets of $X$ and $Y$ which satisfy (*) Let us identify $\mathcal{E} \times \mathcal{F}$ with the family

$$\{E \times F : E \in \mathcal{E}, F \in \mathcal{F}\}$$

of subsets of $X \times Y$. If $E_1 \times F_1, \ldots, E_n \times F_n \in \mathcal{E} \times \mathcal{F}$, then there is a finite partition $\mathcal{P}$ [resp. $\mathcal{Q}$] of $X$ [resp. $Y$] into $\mathcal{E}$-sets [resp. $\mathcal{F}$-sets] which decomposes every $E_i$ [resp. $F_i$]. Now the finite partition $\mathcal{P} \times \mathcal{Q}$ consists of $(\mathcal{E} \times \mathcal{F})$-sets and decomposes every $E_i \times F_i$.

This example extends immediately to Cartesian products of larger (but still finite) collections of sets.

By applying Proposition 2 to these examples, we recover all the cases that we need in 245A of a family of sets being an algebra. However, it takes a little extra work to show that Proposition 2 implies Folland’s abstract Proposition 1.7 itself.

**Exercise 4.** Show that an elementary family of subsets of $X$ (as defined on p23 of Folland) satisfies condition (*).