1 Some applications of the Fubini–Tonelli theorem to $n$-dimensional Lebesgue measure

(Folland Section 2.6)

We have seen that $(\mathbb{R}^n, \mathcal{L}^n, m^n)$ is equal to the completion of the product of $n$ copies of $(\mathbb{R}, \mathcal{L}, m)$. With this in hand, certain facts about $(\mathbb{R}^n, \mathcal{L}^n, m^n)$ become easy consequences of the Fubini–Tonelli theorem for complete spaces.

Lebesgue measure is translation-invariant as an easy consequence of its definition in terms of $h$-rectangles. An important extension of this fact is the rule for transformation Lebesgue measure under an invertible linear change of coordinates, and this takes a little more work to prove.

**Theorem 1.1** (Linear change-of-variables in $\mathbb{R}^n$; Folland 2.44). *Suppose $T$ is an invertible linear map $\mathbb{R}^n \rightarrow \mathbb{R}^n$.*

a. If $f$ is Lebesgue measurable on $\mathbb{R}^n$, so is $f \circ T$. If $f \geq 0$ or $f \in L^1(m)$, then

$$\int f(x) \, dx = |\det T| \int f(T(x)) \, dx. \quad (1)$$

b. If $E \in \mathcal{L}^n$, then $T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det T| m(E)$.

*Proof sketch.* First we prove part a. for a Borel function $f$. To this end, observe that if (1) holds for two invertible linear transformations $S$ and $T$, then it also holds for $S \circ T$, since $|\det(S \circ T)| = |\det S||\det T|$.

Any linear transformation can be written as a composition using three special kinds of transformation: coordinate-scalings, coordinate-swaps, and shears. So it suffices to prove the result in these three special cases. Each is now easy using
the Fubini–Tonelli theorem and previous facts about one-dimensional Lebesgue measure. For instance, if \( n = 2 \) and \( T(x, y) = (x, y + cx) \) (so \( \det T = 1 \)), then

\[
\int f(T(x, y)) \, dm^2(x, y) = \int f(x, y + cx) \, dm^2(x, y) \\
= \int \left[ \int f(x, y + cx) \, dy \right] \, dx \quad \text{(Fubini–Tonelli)} \\
= \int \left[ \int f(x, y) \, dy \right] \, dx \quad \text{(because \( m^1 \) trans-invt.)} \\
= \int f(x, y) \, dm^2(x, y) \quad \text{(Fubini–Tonelli again)}.
\]

Once we have part a. for Borel functions \( f \), we get part b. for Borel sets \( E \) by letting \( f = \chi_E \).

To extend part b. to all Lebesgue sets \( F \), it now suffices to prove that

\[ F \text{ null } \implies T(F) \text{ null}, \]

since Lebesgue measure is the completion of its restriction to the Borel sets. But if \( F \) is null, then there is a null Borel set \( E \supseteq F \). Now applying part b. to \( E \) gives that \( T(E) \supseteq T(F) \) are also both still null sets.

Finally, if \( f : \mathbb{R}^n \to \mathbb{C} \) is Lebesgue measurable, then there is a Borel function \( g : \mathbb{R}^n \to \mathbb{C} \) which agrees with \( f \) a.e. Since the Lebesgue null set \( \{ f = g \} \) is transformed into the Lebesgue null set \( \{ f \circ T = g \circ T \} \), and neither of these can affect the values of any integrals, part a. for \( g \) implies part a. for \( f \). \( \square \)

**Corollary 1.2** (Folland 2.46). *Lebesgue measure is invariant under rotations.*

Theorem 1.1 has some much more versatile generalizations to nonlinear changes of variables. You can find various versions in the literature. Here is Folland’s:

**Theorem 1.3** (Nonlinear change-of-variables in \( \mathbb{R}^n \); Folland 2.47). *Suppose that \( \Omega \) is an open set in \( \mathbb{R}^n \) and \( G : \Omega \to \mathbb{R}^n \) is (i) continuously differentiable, (ii) injective, and (iii) satisfies the condition that \( D_x G \) is invertible for all \( x \in \Omega \).*

a. *If \( f \) is a Lebesgue measurable function on \( G(\Omega) \), then \( f \circ G \) is Lebesgue measurable on \( \Omega \). If \( f \geq 0 \) or \( f \in L^1(G(\Omega), m) \), then*

\[
\int_{G(\Omega)} f(x) \, dx = \int_{\Omega} f(G(x)) |\det D_x G| \, dx.
\]
b. If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and

$$m(G(E)) = \int_E |\det D_s G| \, dx.$$ 

I will sketch the proof of this in class if time permits, but will probably not be able to give many details.

Remark. An important special case of Theorem 1.3 justifies integration using polar coordinates. Folland's Section 2.7 is dedicated to these, but I will not cover them separately in lectures.