1 Reminder: regular Borel measures on $\mathbb{R}$

(From Lec 6, or Folland Sec 1.5)

Characterization of locally finite measures on $\mathcal{B}_{\mathbb{R}}$:

**Theorem 1.1.** If $F : \mathbb{R} \to \mathbb{R}$ is increasing and right-continuous, then there is a unique locally finite Borel measure $\mu_F$ on $\mathcal{B}_{\mathbb{R}}$ such that $\mu_F((a, b]) = F(b) - F(a)$ whenever $a < b$. Every locally finite Borel measure on $\mathbb{R}$ arises this way, and we have $\mu_F = \mu_G$ iff $F - G$ is constant.

2 Lebesgue differentiation in one dimension

Using Theorem 1.1, we can interpret the Lebesgue differentiation theorems in one dimension in terms of the functions $F$. The result is actually closer to Lebesgue’s original version of the ‘differentiation theorem’.

**Theorem 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be increasing, and let $G(x) = F(x^+)$.

a. The set of points at which $F$ is discontinuous is countable.

b. $F$ and $G$ are differentiable a.e., and $F' = G'$ a.e.

*Ideas.*

1. The open intervals $(F(x^-), F(x^+))$ $(x \in \mathbb{R})$ are all disjoint, so at most countably many can be nonempty.

2. Differences of $G$-values can be expressed in terms of $\mu_G$. Using this, $G'$ is given a.e. by the Lebesgue differentiation theorem.
3. If $H = G - F$, then $H$ is nonzero at only countably many points. Enumerate them, and make the corresponding sum of point masses. Applying Lebesgue differentiation to that sum gives $H' = 0$ a.e.

\[\square\]

3 Signed measures and functions of BV

Now let $\nu$ be a locally finite signed measure on $\mathbb{R}$. Then we can still look for a function $F$ which satisfies

$$\nu((a,b]) = F(b) - F(a),$$

but now $F$ need not be increasing, since $\nu$ may take negative values.

Jordan decomposition gives $\nu = \nu^+ - \nu^-$, and we already know we can write $\nu^\pm = \mu_{F^\pm}$, so

$$\nu((a,b]) = \nu^+((a,b]) - \nu^-((a,b]) = F(b) - F(a) \quad \text{where } F = F^+ - F^-.$$

So now the relevant functions $F$ for representing signed measures are differences of right-continuous increasing functions.

But a given function $F$ may be expressible as a difference of increasing functions in many different ways. On the other hand, finding one such expression in the first place may be difficult.

Next task: a more ‘intrinsic’ characterization of which functions may be expressed as above.

**Warning:** Folland tells the whole story for complex measures. But I’m going to restrict attention to finite signed measures — the ideas and outcomes are all essentially the same, but the explanation is a little simpler. There’s also a generalization to infinite but locally finite signed measures, but I’ll omit that too.

**Definition 3.1.** Given a function $F : \mathbb{R} \to \mathbb{R}$, its total variation function $T_F : \mathbb{R} \to [0, \infty]$. A function $F$ having bounded variation; the space $BV$. Total variation on a closed bounded interval.

**Example 3.2** (Folland 3.25).

1. Bounded and increasing $\implies BV$, and in fact $T_F(x) = F(x) - F(-\infty)$.

2. If $F, G \in BV$ and $a, b \in \mathbb{R}$ then $aF + bG \in BV$. 

3. If $F$ is differentiable on $\mathbb{R}$ and $F'$ is bounded, then $F$ has bounded variation over any bounded interval.

4. The function $F(x) = \sin x$ has BV over any bounded interval, but not over the whole of $\mathbb{R}$.

5. The function
   $$F(x) := \begin{cases} x \sin(x^{-1}) & x > 0 \\ 0 & x \leq 0 \end{cases}$$
   is continuous, but does not have BV over $[a, b]$ for any $a \leq 0 < b$.

**Lemma 3.3** (Folland 3.26). If $F \in BV$, then $T_F + F$ and $T_F - F$ are increasing.

**Theorem 3.4** (Key part of Folland 3.27 for real-valued functions). If $F : \mathbb{R} \rightarrow \mathbb{R}$, then $F \in BV$ iff $F$ is the difference of two bounded increasing functions. If $F \in BV$ then one such decomposition is
   $$\frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F).$$
   We call $\frac{1}{2}(T_F + F)$ (resp. $\frac{1}{2}(T_F - F)$) the **positive** (resp. **negative** variation) of $F$, and their combination as above is the **Jordan decomposition of $F$**. As far as I know, this decomposition is actually due to Jordan, and motivates the name 'Jordan decomposition' for our previous result about signed measures: see Folland’s exercise 3.29 for the connection.

**Corollary 3.5** (Remainder of Folland 3.27 for real-valued functions). If $F \in BV$, then
   1. $F(x+)$ and $F(x-)$ exist for every $x$, and so do $F(\pm \infty)$;
   2. the set of points at which $F$ is discontinuous is countable;
   3. if $G(x) = F(x+)$, then $F'$ and $G'$ exist and are equal a.e.

If $\nu$ and $F$ are related as in (1), then $F$ is determined only up to an arbitrary additive constant. The next definition removes this issue.

**Definition 3.6.** **Normalized** elements of $BV$; the space $NBV$. Observe that if $F \in BV$ and is right-continuous, then
   $$F(x) - F(-\infty) \in NBV.$$
We next check that Jordan decomposition preserves right-continuity.

**Lemma 3.7** (Folland 3.28). If \( F \in BV \), then \( T_F(-\infty) = 0 \). If \( F \) is also right-continuous, then so is \( T_F \).

Putting the previous results together, we now have a precise correspondence between finite signed Borel measures and NBV functions. This is the ‘signed’ generalization of our earlier correspondence between finite positive measures and bounded increasing functions.

**Theorem 3.8** (Folland 3.29 for finite signed measures). If \( \mu \) is a finite signed Borel measure on \( \mathbb{R} \) and \( F(x) = \mu(-\infty, x] \), then \( F \in NBV \). Conversely, if \( F \in NBV \), then there is a unique finite signed Borel measure \( \mu_F \) such that \( F(x) = \mu_F(-\infty, x] \), and \( |\mu_F| = \mu_{T_F} \) (last part is a homework exercise).