**Problem 1**

In the $e^+e^-$ collisions, assuming the center-of-mass energy is $\sqrt{s}$, please calculate the value of $R$ as defined below

$$R = 3 \times \left( \frac{2}{3} \right)^2$$

plot your calculated $R$ as a function of $\sqrt{s}$. What is the minimum $\sqrt{s}$ to produce a bottom quark in such collisions?

<table>
<thead>
<tr>
<th>Value of $R$</th>
<th>$2m_t c^2$</th>
<th>$\sqrt{s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{4}{3}$</td>
<td>2.44 MeV</td>
<td>$\sqrt{s} &lt; 2m_d c^2 = 9.4$ MeV</td>
</tr>
<tr>
<td>$\frac{5}{3}$</td>
<td>5.44 MeV</td>
<td>$\sqrt{s} &lt; 2m_d c^2 = 192$ MeV</td>
</tr>
<tr>
<td>$\frac{10}{3}$</td>
<td>2.54 GeV</td>
<td>$\sqrt{s} &lt; 2m_t c^2 = 9.4$ GeV</td>
</tr>
<tr>
<td>$\frac{11}{3}$</td>
<td>4.44 GeV</td>
<td>$\sqrt{s} &lt; 2m_b c^2 = 346$ GeV</td>
</tr>
</tbody>
</table>

The minimum $\sqrt{s}$ to produce a bottom quark is $2m_b c^2 \approx 9.4$ GeV.
Problem 2

$F_2^{ep}(x)$ and $F_2^{en}(x)$ are the structure functions of deep inelastic scattering for an electron on a proton or neutron target, respectively. Please derive the following relation,

$$\int_0^1 \frac{dx}{x} [F_2^{ep}(x) - F_2^{en}(x)] = \frac{1}{3} + \frac{2}{3} \int_0^1 dx \left[ \bar{u}(x) - \bar{d}(x) \right],$$

where the quark distributions refer to the proton. For your reference: such a relation is called the Gottfried sum rule.

$$F_2^{ep}(x) = \sum_{q=u,d,s} e_q^2 [q(x) + \bar{q}_q(x)]$$

$$= x \left( e_u^2 [u_u^p(x) + u_s^p(x) + \bar{u}_u(x)] + e_d^2 [d_d^p(x) + d_s^p(x) + \bar{d}_d(x)] + e_s^2 [s_s^p(x) + \bar{s}_s(x)] \right)$$

$$= x \left( \frac{4}{9} [u_u^p(x) + u_s^p(x) + d_d^p(x)] + \frac{1}{9} [d_d^p(x) + d_s^p(x) + \bar{d}_d(x)] + \frac{1}{9} [s_s^p(x) + \bar{s}_s(x)] \right)$$

$$F_2^{en}(x) = x \left( \frac{4}{9} [u_u^p(x) + u_s^p(x) + \bar{u}_u(x)] + \frac{1}{9} [d_d^p(x) + d_s^p(x) + \bar{d}_d(x)] + \frac{1}{9} [s_s^p(x) + \bar{s}_s(x)] \right)$$

Using isospin symmetry, we have

$$u_u^p(x) = d_d^p(x) = u_v(x)$$

$$d_d^p(x) = u_u^p(x) = d_v(x)$$

$$u_s^p(x) = d_u^p(x) = u_s(x)$$

$$d_s^p(x) = u_d^p(x) = d_s(x)$$

The effective quantum number of sea quarks cancel out. Therefore, $q_s(x) = \bar{q}_s(x)$. Also, we assume the distribution of s-quarks in protons and neutrons to be identical, that is, $s^p_s = s^n_s$. Thus,

$$F_2^{ep}(x) = x \left( \frac{4}{9} [u_v(x) + u_s(x) + \bar{u}_u(x)] + \frac{1}{9} [d_v(x) + d_s(x) + \bar{d}_d(x)] + \frac{1}{9} [s_s^p(x) + \bar{s}_s^p(x)] \right)$$

$$F_2^{en}(x) = x \left( \frac{1}{9} [u_v(x) + u_s(x) + \bar{u}_u(x)] + \frac{4}{9} [d_v(x) + d_s(x) + \bar{d}_d(x)] + \frac{1}{9} [s_s^n(x) + \bar{s}_s^n(x)] \right)$$

$$F_2^{ep}(x) - F_2^{en}(x) = x \left( \frac{1}{3} [u_v(x) - d_v(x)] + \frac{2}{3} [\bar{u}_u(x) - \bar{d}_d(x)] \right)$$

$$\int_0^1 \frac{dx}{x} [F_2^{ep}(x) - F_2^{en}(x)] = \int_0^1 \frac{dx}{x} \left( \frac{1}{3} [u_v(x) - d_v(x)] + \frac{2}{3} [\bar{u}_u(x) - \bar{d}_d(x)] \right)$$

$$= \frac{1}{3} \left[ \int_0^1 dx u_v(x) - \int_0^1 dx d_v(x) \right] + \frac{2}{3} \int_0^1 dx \left[ \bar{u}_u(x) - \bar{d}_d(x) \right]$$

$$= \frac{1}{3} [2 - 1] + \frac{2}{3} \int_0^1 dx \left[ \bar{u}_u(x) - \bar{d}_d(x) \right]$$

$$= \frac{1}{3} + \frac{2}{3} \int_0^1 dx \left[ \bar{u}_u(x) - \bar{d}_d(x) \right].$$
Problem 3

The distributions of valence quarks \( q_v(x) \), sea quarks \( q_s(x) \), and the gluons \( g(x) \) in the proton are given by

\[
q_v(x) = A \frac{(1-x)^3}{\sqrt{x}}, \quad q_s(x) = B \frac{(1-x)^6}{x}, \quad g(x) = 4 \frac{(1-x)^6}{x}.
\]

Please finish the following questions.
(a) What are the meanings of the following integrals?

\[
\int_0^1 dx \, q_v(x), \quad \int_0^1 dx \, q_s(x), \quad \int_0^1 dx \, g(x).
\]

\( \int_0^1 dx \, q_v(x) \) is the number of valence quarks in a proton.
\( \int_0^1 dx \, q_s(x) \) is the number of sea quarks in a proton.
\( \int_0^1 dx \, g(x) \) is the number of gluons in a proton.

(b) Determine the value of \( A \). Hint: there are three valence quarks in the proton.

\[
\int_0^1 dx \, q_v(x) = \int_0^1 dx \, A \frac{(1-x)^3}{\sqrt{x}} = 3
\]
\[
A \int_0^1 dx \left[ \frac{1-3x+3x^2-x^3}{\sqrt{x}} \right] = 3
\]
\[
A \int_0^1 dx \left[ x^{-1/2} - 3x^{1/2} + 3x^{3/2} - x^{5/2} \right] = 3
\]
\[
A \left[ 2x^{1/2} + 2x^{3/2} + \frac{6}{5}x^{5/2} - \frac{2}{7}x^{7/2} \right]_0^1 = 3
\]
\[
A \left[ 2 - 2 + \frac{6}{5} - \frac{2}{7} \right] = A \left( \frac{32}{35} \right) = 3
\]
\[
A = \frac{105}{32}
\]

(c) What is the meaning of the following integral?

\[
\int_0^1 dx \, [q_v(x) + q_s(x) + g(x)].
\]

This integral is the total momentum fraction of the proton carried by valence quarks, sea quarks, and gluons.
(d) Determine the value of B.

Since the proton is entirely composed of valence quarks, sea quarks, and gluons, the above integral should be equal to one.

\[
\int_0^1 dx \left[ q_v(x) + q_s(x) + g(x) \right] = 1
\]

\[
\int_0^1 dx \left[ A \left( \frac{1-x}{\sqrt{x}} \right)^3 + B \left( \frac{1-x}{x} \right)^8 + 4 \left( \frac{1-x}{x} \right)^6 \right] = 1
\]

\[
A \left[ \frac{2}{3} x^{2/3} - \frac{6}{5} x^{5/2} + \frac{6}{7} x^{7/2} - \frac{2}{9} x^{9/2} \right]_0^1 + B \left[ \frac{1}{9} - \frac{2}{7} x^2 \right]_0^1 + 4 \left[ \frac{1}{7} - \frac{2}{7} x^3 \right]_0^1 = 1
\]

\[
A \left[ \frac{32}{315} \right] + B \left[ \frac{9}{9} + \frac{4}{7} \right] = 1
\]

\[
B = \frac{6}{7}
\]

(e) What is the meaning of this integral?

\[
\int_{x_{\text{min}}}^1 \! dx \left[ q_v(x) + q_s(x) + g(x) \right]
\]

Calculate the numerical value of the above integral when \(x_{\text{min}} = 0.4\).

This integral is the number of valence quarks, sea quarks, and gluons carrying a momentum fraction of the proton between \(x_{\text{min}}\) and one.

\[
\int_{0.4}^1 \! dx \left[ q_v(x) + q_s(x) + g(x) \right] = \int_{0.4}^1 \! dx \left[ A \left( \frac{1-x}{\sqrt{x}} \right)^3 + B \left( \frac{1-x}{x} \right)^8 + 4 \left( \frac{1-x}{x} \right)^6 \right]
\]

\[
= A \left[ 2x^{1/2} + 2x^{3/2} + \frac{6}{5} x^{5/2} - \frac{2}{7} x^{7/2} \right]_{0.4}^1 + B \left[ \frac{1}{x} - 8 + 28x - 56x^2 + 70x^3 - 56x^4 + 28x^5 - 8x^6 \right]_{0.4}^1 + 4 \left[ \frac{1}{x} - 6 + 15x - 20x^2 + 15x^3 - 6x^4 + x^5 \right]_{0.4}^1
\]

\[
= A \left[ 2x^{1/2} + 2x^{3/2} + \frac{6}{5} x^{5/2} - \frac{2}{7} x^{7/2} \right]_{0.4}^1 + B \left[ \log x - 8x + 14x^2 - \frac{56}{3} x^3 + \frac{35}{2} x^4 - \frac{56}{5} x^5 + \frac{14}{3} x^6 - \frac{8}{7} x^7 + \frac{x^8}{8} \right]_{0.4}^1 + 4 \left[ \log x - 6x + \frac{15}{2} x^2 - \frac{20}{3} x^3 + \frac{15}{4} x^4 - \frac{6}{5} x^5 + \frac{x^6}{6} \right]_{0.4}^1
\]

\[
= A \times .0455 + B \times .000246 + 4 \times .00856
\]

\[
= .186
\]