Exercise 1. Let $\Omega$ be a finite sample space. Let $f : \Omega \to [0,1]$ be a function that sums up to 1, i.e., $\sum_{x \in \Omega} f(x) = 1$. Let $2^\Omega$ denote the set of all subsets of $\Omega$. Define a function $G : 2^\Omega \to [0,1]$ by

$$G(E) = \sum_{x \in E} f(x).$$

Show that $G$ is a probability measure on $\Omega$.

Exercise 2. Let $(\Omega, \mathbb{P})$ be a probability space and let $A \subseteq \Omega$ be an event. Show that $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Exercise 3 (Roll of three dice). Suppose we roll three dice and all possible joint outcomes are equally likely. Identify the sample space $\Omega = \{1,2,3,4,5,6\}^3$ as a $(6 \times 6 \times 6)$ 3-dimensional integer lattice, and let $X$, $Y$, and $Z$ denote the outcome of each die.

(i) Write down the probability distribution on $\Omega$.

(ii) For each $k \geq 1$, show that

$$\mathbb{P}(X + Y + Z = k) = \frac{\text{# of intersections between the plane } x + y + z = k \text{ and } \Omega}{6^3}.$$ 

What are the minimum and maximum possible values for $X + Y + Z$?

(iii) Draw a cube for $\Omega$ and planes $x + y + z = k$ for $k = 3,5,10,11,16,18$. Argue that the intersection gets larger as $k$ increases from 3 to 10 and smaller as $k$ goes from 11 to 18. Conclude that 10 and 11 are the most probable values for $X + Y + Z$.

(iv) Consider the following identity

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$$

$$= x^{18} + 3x^{17} + 6x^{16} + 10x^{15} + 15x^{14} + 21x^{13} + 25x^{12} + 27x^{11} + 27x^{10}$$

$$+ 25x^9 + 21x^8 + 15x^7 + 10x^6 + 6x^5 + 3x^4 + x^3$$

Show that the coefficient of $x^k$ in the right hand side equals the size of the intersection between $\Omega$ and the plane $x + y + z = k$. Conclude that

$$\mathbb{P}(X + Y + Z = 10) = \mathbb{P}(X + Y + Z = 11) = \frac{27}{6^3} = \frac{1}{8}.$$ 

(This way of calculating probabilities is called the generating function method.)

Exercise 4. Suppose Nate commutes to campus by Bruin bus, which arrives at his nearby bus stop every 10 min. Suppose each bus waits at the stop for 1 min. What is the probability that Nate takes no more than 3 min at the stop until he takes a bus? (Hint: Represent the sample space as a unit square in the coordinate plane)