Exercise 1 (Laplace's rule of succession). Laplace 'computed' the probability that the sun will rise tomorrow, given that it has risen for the preceding 5000 years. The combinatorial model is as follow. Suppose we have \( N \) different coins, where the \( k \)th coin has probability \( k/N \) of coming up heads. We choose one of the \( N \) coin uniformly at random and flip it \( n \) times. For each \( 1 \leq i \leq n \), let \( R_i \) be the event that the \( i \)th flip comes up heads. We are interested in the following conditional probability

\[
P(R_{n+1} | R_n \cap R_{n-1} \cap \cdots \cap R_1). \tag{1}
\]

If we think of coin coming up heads as the event of sun rising, then this is a model for the probability that the sun will rise tomorrow given that it has risen for the past \( n \) days.

(i) Write \( R'_n = R_n \cap R_{n-1} \cap \cdots \cap R_1 \). Show that

\[
P(R_{n+1} | R'_n) = \frac{P(R_{n+1} \cap R'_n)}{P(R'_n)} = \frac{P(R_{n+1})}{P(R'_n)} . \tag{2}
\]

(ii) For each \( 1 \leq i \leq n + 1 \), use partitioning to show that

\[
P(R'_i) = \sum_{k=1}^{N} P(R'_i | \text{prob. of heads is } k/N)P(\text{prob. of heads is } k/N) = \sum_{k=1}^{N} \left( \frac{k}{N} \right)^i \frac{1}{N}. \tag{3}
\]

(iii) By considering the upper and lower Riemann sums we have

\[
\sum_{k=1}^{N} \left( \frac{k}{N} \right)^i \frac{1}{N} \leq \int_0^1 t^i dt \leq \sum_{k=0}^{N-1} \left( \frac{k}{N} \right)^i \frac{1}{N}. \tag{4}
\]

Using (ii) and (iii), show that

\[
P(R'_i) \approx \int_0^1 t^i dt \approx \frac{1}{i+1}, \tag{5}
\]

where the approximation becomes exact as \( N \to \infty \).

(iv) From (i), conclude that

\[
P(R_{n+1} | R'_n) \approx \frac{n+1}{n+2}. \tag{6}
\]

For \( n = 5000 \times 365 \) (days), we have \( P(R_{n+1} | R'_n) \approx 0.9999994520 \). So it’s pretty likely that the sun will rise tomorrow as well.

Exercise 2 (Refer to Ex 4.5 in note1). Suppose we have a prior distribution \( \pi = [0.0025, 0.3680, 0.6305] \) on the sample space \( \Omega = \{0.2, 0.5, 0.8\} \) for the inference problem of unknown parameter \( \Theta \). Suppose we are given the data that two heads come up from ten independent flips of probability \( \Theta \) coin. Compute the posterior distribution using this data and Bayesian inference.

Exercise 3. A test for pancreatic cancer is assumed to be correct %95 of the time: if a person has the cancer, the test results in positive with probability 0.95, and if the person does not have the cancer, then the test results in negative with probability 0.95. From a recent medical research, it is known that only %0.05 of the population have pancreatic cancer. Given that the person just tested positive, what is the probability of having the cancer?