These are really sketchy notes just to help you fix typos . . .
MODELLING CHANGE

Today: A mathematical model of a dynamical system describes how the values of the state variables change over time.

**Example 1**: Bathtub, faucet, drain. (Tap)

- We will model with one state variable

  \[ W = \text{amount of water} \quad \text{(liters, L)} \]

  - in the tub

- Assumptions ...
  
  Only two things happen

  - Faucet pours water into the tub
  - Drain drains water out of the tub

**Notation**

- Let \( W(t) \) = amount of water in tub (L) at time \( t \)
- \( W'(t) \) = rate at which amount of water in the tub is changing at time \( t \) (L/min).
EXAMPLE USING NOTATION

Suppose at \( t = 2 \) mins,

- faucet and drain are closed
  \[ x'(2) = 0 \quad (L/min) \]
- + 10
- - 5
- + 10 - 5.

Main goal of math. modelling (in this example)

- Describe \( W'(t) \) in terms of \( W(t) \)
- i.e. describe how changes in the system depend on the current state.

Might not even depend on \( W(t) \) in a very complicated way:

If faucet and drain are always closed,

\[ W'(t) = 0 \quad (L/min) \]

depends on \( W(t) \) in such a silly way it feels weird to say.
The previous examples suggest a good way to approach setting up a model:

\[ W'(t) = + \text{(rate at which faucet is pouring at } t) - \text{(rate drain draining)} \]

\[ W'(t) = I(t) - O(t) \]

This is something we'll often try to do when setting up a mathematical model. There might be more state variables though.

More assumptions...

Faucet pouring at constant rate of \( f = 10 \text{ L/min} \).

\[ I(t) = f = 10 \text{ L/min} \]

The drain is draining at a rate proportional to the amount of water in the tub:

\[ O(t) = k \cdot W(t) \text{ L/min} \]

Proportion const. depends on size of drain.

Maybe \( k = 0.2 \text{ L/min} \).
SUMMARY OF EXAMPLE 1

\[ W'(t) = I(t) - O(t) \]

\[ W'(t) = 10 - 0.2W(t) \]

**Called \underline{CHANGE EQUATION}**

STATE VARIABLE \( W = W(t) \) \( \rightarrow \) \underline{CHANGE UNDERGOES}

TWO PARAMETERS \( f = 10 \) \( \text{L/MIN} \)
\( \kappa = 0.2 \) \( \text{L/MIN} \) \( \leftarrow \) \underline{CONSTANTS}

THINKING ABOUT WHAT THE MODEL SAYS A LITTLE:

• SUPPOSE AT TIME \( t = 0 \) MIN, BATH IS EMPTY.
  
  IN MATH., \( W(0) = 0 \).

• \[ \star \] SAYS \( W'(0) = 10 - 0.2W(0) = 10 \) \( \text{L/MIN} \)

RIGHT AT THE BEGINNING, THE AMOUNT OF WATER IN TUB IS INCREASING AT RATE OF 10 L/MIN.
Example 2

Assumption 1.

- Imagine you're a genderless tuna capable of asexual reproduction.
- For now, assume no sharks.
- Steady supply of food.
  As much scale as needed in sea. Immortal.
- Completely unique: Think the same as all other tuna.

One question on your mind:
  How many baby tuna should I have each year?

Answer: b per yr.

b is called per capita birth rate of tuna.

b is the number of tuna born per year per tuna.

Units? (tuna/yr)/tuna = 1/yr

Use whichever sounds better to you.
ASSUMPTION 2.

- IMAGINE YOU’RE A SHARK.
- FOR NOW, ASSUME NO TUNA.
- TUNA IS THE ONLY FOOD YOU LIKE, SO YOU ARE DEPRESSED AND HAVE NO DESIRE TO REPRODUCE.
- ONLY ONE Q ON YOUR MIND: WHEN WILL I DIE?

ANSWER: IT SEEMS LIKE EACH YEAR, THE FRACTION OF ALL SHARKS THAT DIE IS SOME NUMBER D (SHARKS PER SHARK) PER YEAR.

D IS THE VEL CAPITA DEATH RATE OF SHARKS.

D IS THE NUMBER OF SHARK DEATHS PER YEAR PER SHARK.

UNITS? (SHARK/YR)/SHARK = 1/YR.
Let $S(t) =$ SHARK POPULATION AT TIME $t$.
$T(t) =$ TUNA POPULATION AT TIME $t$.

Last time, introduced some notation:

$S'(t) =$ RATE OF CHANGE OF SHARK POPULATION AT TIME $t$.
$T'(t) =$ RATE OF CHANGE OF TUNA POPULATION AT TIME $t$.

Based on our first 2 assumptions:

$S'(t) = -a S(t)$

- Units: Sharks /Year
- $a$ need this
- These are different

\[\text{(should go down, i.e., negative rate of change)}\]
$\alpha = 0.17 \text{ 1/yr}$

We could say:

- the *per capita* rate of death is $0.17 \text{ 1/yr}$.
- the death rate is proportional to $S$ with proportionality constant $0.17 \text{ 1/yr}$.

Similarly,

$$T'(t) = \alpha T(t).$$

Currently, there is no interaction between $S$ and $T$.

Last time, said good to study inflow and outflow:

$$S'(t) = \boxed{I_S(t)} - \boxed{O_S(t)}$$

$$T'(t) = \boxed{I_T(t)} - \boxed{O_T(t)}$$

We've dealt with the boxed terms.

What about the others?
ASSUMPTION 3:

LET'S MESS WITH THE TUNA'S WORLD BY INTRODUCING SHARKS.

NOW WE'LL ASSUME THAT THE ONLY WAY TUNA CAN DIE IS BY BEING EATEN BY SHARKS.

IF YOU'RE A TUNA, A NEW QUESTION:

HOW LIKELY IS IT THAT I'LL BE EATEN BY A SHARK IN THE NEXT YEAR?

WHAT FRACTION OF THE TUNA POPULATION WILL BE EATEN IN THE NEXT YEAR?

A: SHOULD BE PROPORTIONAL TO NUMBER OF SHARKS S.

\[ \beta S \]

\[ \text{THIS FRACTION OF TUNA WILL BE EATEN EACH YEAR} \]

\[ \text{UNITS OF } \beta S \text{ IS } 1/\text{YEAR} \]

\[ \text{UNITS OF } \beta \text{ IS } 1/\text{SHARK, YEAR} \]

\[ \text{THIS KINDA WEIRD, DON'T DWELL ON IT.} \]

\[ \text{THIS GIVES } 0(t) = \beta S t. \]
• The per capita death rate of tuna is proportional to the number of sharks with proportion constant $\beta$.

• The rate at which tuna are eaten by sharks is proportional to the product of the populations with prop. constant $\beta$.

Assumption 4:

Now tuna are around sharks are happy and reproduce.

Q: How often should I have a baby shark.

A: Should be proportional to amount of food, i.e. amount of tuna

Think of as one quantity for now

$S(t) = m\beta ST.$
• PEL CAPITAL REPRODUCTION RATE OF SHARK IS PROPORTIONAL TO NUMBER OF TUNA WITH PROPORTION CONSTANT $m_f$.

• THE RATE AT WHICH SHARKS REPRODUCE IS PROPORTIONAL TO THE PRODUCT OF THE POPULATIONS WITH PROD. CONSTANT $m_f$.

\[ s' = m_f st - a s \]

\[ t' = b t - bst \]
EXAMPLE 1

\[ W' = 10 - 0.2W \quad (L/\text{min}) \]

**Example 1**

1. **Incoming Flow Term**
   - Per capita birth rate of tuna \( b \).

2. **Outflow Term**
   - Per capita death rate of shark \( d \).

3. **Outflow Term**
   - Per capita death rate of tuna (proportional to \( S \) and \( W \)) proportional constant \( \beta \).

4. **Inflow Term**
THING TO NOTE

ST is an interaction term.

Measures how frequently S,T meet.

\[ S \times T \propto \text{Proportional to Tuna Death Rate} \]
\[ \text{} \propto \text{Shark Birth Rate} \]

(m controls "how much food a shark thinks it needs to raise another child").

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Sharks \(\rightarrow\) Foxes
Tuna \(\rightarrow\) Rabbits

Pick some numbers.

\[ F' = 0.01 FR - 0.1F \]
\[ R' = 0.3 R - 0.02RF \]
WHY I SCREAMED:

PEL CAPITA DEATH RATE OF TUNA IS $\beta_S$
DEATH RATE OF TUNA IS $\beta_{ST}$.

WRITING "PEL CAPITA" HERE IS A NO-NO.

PEL CAPITA BIRTH RATE OF SHARK IS $m\beta_T$
BIRTH RATE OF SHARK IS $m\beta_{ST}$.

WRITING "PEL CAPITA" HERE IS A NO-NO.