1 Example

Prove that if $L_1$ and $L_2$ are regular, then $L_1 \cup L_2$ is regular.

Proof. Let $L_1$ and $L_2$ be regular.
Then, there exist DFAs

\[ M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \]
\[ M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \]

such that $L(M_1) = L_1$ and $L(M_2) = L_2$.

Idea:
We will construct an NFA to recognize $L_1 \cup L_2$ by creating a new starting state and then having $\epsilon$-transitions from this new starting state to the starting states of $M_1$ and $M_2$. N can then nondeterministically choose to follow either $M_1$ or $M_2$. 

![Diagram of NFAs M1 and M2 with epsilon transitions to q0]

1 Proof

CS 181 Proof Example

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More formally,
Let $N = (Q', \Sigma, \delta', q_0, F')$ be an NFA such that

$$\begin{align*}
Q' &= Q_1 \cup Q_2 \cup \{q_0\} \\
F' &= F_1 \cup F_2
\end{align*}$$

For $q \in Q', a \in \Sigma \cup \{\epsilon\}$,

$$\delta'(q, a) = \begin{cases} \\
\delta_1(q, a) & \text{if } q \in Q_1, a \neq \epsilon \\
\delta_2(q, a) & \text{if } q \in Q_2, a \neq \epsilon \\
\{q_1, q_2\} & \text{if } q = q_0, a = \epsilon \\
\emptyset & \text{else}
\end{cases}$$

Then, $L(N) = L_1 \cup L_2$ since

$L(N) \supseteq L_1 \cup L_2$ (i.e. if $x \in L_1 \cup L_2$, then $x \in L(N)$).
Let $x \in L_1 \cup L_2$. Then, $x \in L_1$ or $x \in L_2$.
Suppose $x \in L_1$. Then, $M_1(x)$ accepts. Thus, on input $x$, $M_1$ goes from the starting state $q_1$ to a state in $F_1$. Therefore, on input $x$, $N$ can first go from state $q_0$ to $q_1$ on $\epsilon$. Then, since $N$ has the same transitions as $M_1$ when in states in $Q_1$, $N$ can also go from state $q_1$ to a state in $F_1 \subseteq F'$ on input $x$. Thus, $N(x)$ accepts. Similarly, if $x \in L_2$, then $N(x)$ can go from $q_0$ to $q_2$ on input $\epsilon$ and then from $q_2$ to a state in $F_2 \in F'$ on input $x$. Therefore, $x \in L(N)$.

$L(N) \subseteq L_1 \cup L_2$ (i.e. if $x \in L(N)$, then $x \in L_1 \cup L_2$).
Let $x \in L(N)$. Then, on input $x$, $N$ goes from state $q_0$ to a state $q' \in F'$. Since $q_0$ is not in $F'$, then $N$ must first transition out of $q_0$. But, the only transitions from $q_0$ are the $\epsilon$-transitions to either $q_1$ or $q_2$. Suppose, without loss of generality, that $N$ takes the transition to $q_1$. Then, on $x$, $N$ goes from $q_1$ to a state $q' \in F'$. However, since $N$ follows the transitions of $M_1$ when in states of $Q_1$, then the only states that $N$ can reach from $q_1$ are states in $M_1$. Thus, $q' \in F' \cap Q_1 = F_1$. Therefore, on $x$, $N$ goes from $q_1$ to a state in $F_1$ by following the same transitions and states as in $M_1$. But this means that $M_1(x)$ accepts. Now, suppose that $N$ instead first transitioned from $q_0$ to $q_2$ on $\epsilon$. By a similar argument, $M_2(x)$ accepts. Thus, $x \in L_1 \cup L_2$. \[\blacksquare\]