Regular Languages

1  \textit{DROPOUT}

(1.33 in textbook)
Prove that if \( L \) is regular, then \( DROPOUT(L) = \{xz | xyz \in L, \text{ where } x,y,z \in \Sigma^*, |y| = 1\} \) is regular.

\textbf{Proof. Intuition:}
If \( M \) is a DFA for \( L \), then we will make a new NFA \( N \) that has two copies of \( M \): \( M_1 \) and \( M_2 \), representing \( x \) and \( z \) respectively. We will start at \( M_1 \), representing transitioning on \( x \). At any point along the computation, we can simulate the missing \( y \) by epsilon transitioning from \( M_1 \) to the equivalent next state in \( M_2 \) as if we had just received some letter \( y \). Then, we can finish computing in \( M_2 \), representing transitioning on \( z \). We accept if we end up in an accepting state of \( M_2 \).
Construction:

Let $L$ be a regular language. Then, there exists a DFA $M = (Q, \Sigma, \delta, q_{\text{start}}, F)$ such that $L(M) = L$.

Let us consider two copies of $M : M_1 = (Q_1, \Sigma, \delta_1, q_{1,\text{start}}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{2,\text{start}}, F_2)$

Let $q_i$ and $q_{2i}$ represent the corresponding states in $M_1$ and $M_2$ of state $q_i$ in $M$.

Let $N = (Q', \Sigma, \delta', q_0', F')$ be an NFA such that

\[
Q' = Q_1 \cup Q_2
\]
\[
q_0' = q_{1,\text{start}}
\]
\[
F' = F_2
\]
\[
\delta'(q, a) = \begin{cases} 
\delta_1(q, a) & \text{if } q \in Q_1, a \neq \epsilon \\
\delta_2(q, a) & \text{if } q \in Q_2, a \neq \epsilon \\
\{p \mid p \text{ is the corresponding state in } Q_2 \text{ of } \delta_1(q,b) \text{ for some } b \in \Sigma \} & \text{if } q \in Q_1, a = \epsilon \\
\emptyset & \text{else}
\end{cases}
\]

Now, if $w = xz \in DROPOUT(L)$, then for some $y \in \Sigma$, then $xyz \in L$, so $M(xyz)$ accepts. Thus, $M$ goes from $q_{\text{start}}$ to some state $q_a$ on $x$, from $q_a$ to some $q_b$ on $y$, and from $q_b$ to some $q_f \in F$ on $z$. Therefore, $N$ can go from $q_0' = q_{1,\text{start}}$ to $q_{1a}$ on $x$ since $M_1$ is a copy of $M$. Then, $N$ can epsilon transition from $q_{1a}$ to $q_{2b}$ since $q_{2b}$ is the corresponding state in $M_2$ of $q_{1b} = \delta_1(q_{1a}, y)$. Then, $N$ can transition from $q_{2b}$ to $q_{2f}$ on input $z$ since $M_2$ is a copy of $M$. Therefore, since $q_{2f} \in F_2 = F'$, then $N(xz) = N(w)$ accepts.

If $w \in L(N)$, consider the path taken by $N$ on input $w$. Since we start in $M_1$ and the only accepting states are in $M_2$, then we must transition at some point from $M_1$ to $M_2$. Since the only transitions between $M_1$ and $M_2$ are the one-way epsilon transitions from $M_1$ to $M_2$, then we must take exactly one of these transitions. So, on input $w$, $N$ must go from $q_0' = q_{1,\text{start}}$ to some $q_{1a}$, then take an epsilon transition from $q_{1a}$ to some $q_{2b}$, and then transition from $q_{2b}$ to some state $q_{2f} \in F_2 = F'$. Let $x$ represent the part of $w$ that takes $N$ from $q_0' = q_{1,\text{start}}$ to $q_{1a}$ and $z$ represent the part of $w$ that takes $N$ from $q_{2b}$ to some state $q_{2f}$. Then, $w = xz$. But, since $M_1$ and $M_2$ are copies of $M$ and epsilon-transitions between $q_{1a}$ to $q_{2b}$ only occur when there is a corresponding transition $\delta(q_a, y) = q_b$ for some $y$, then this means that $M$ can go from $q_{\text{start}}$ to some state $q_a$ on $x$, from $q_a$ to some $q_b$ on $y$, and from $q_b$ to some $q_f \in F$ on $z$. Since $M$ is deterministic, this is the only path that $M$ can take on $xyz$. Thus, $M(xyz)$ accepts, so $xyz \in L$. But this means that $w = xz \in DROPOUT(L)$. ■
**Pumping Lemma**

2. \( L_{exp} = \{0^{2^k} \mid k \geq 0 \} \)

(1.29c in textbook)

Prove that \( L_{exp} = \{0^{2^k} \mid k \geq 0 \} \) is not regular.

\textbf{Proof.} Suppose, for sake of contradiction, that \( L_{exp} \) is regular.
By the pumping lemma, \( \exists \) a pumping length \( n \geq 0 \). Consider \( x = 0^{2^n} \in L_{exp} \).
By the pumping lemma, since \( |x| \geq n \), there exists strings \( a, b, c \) such that \( x = abc \), \( 0 < |b| \leq n \), \( |ab| \leq n \), and \( \forall i \geq 0, \ ab^i c \in L_{exp} \). Since \( x \) consists only of 0’s, then \( a,b,c \) consist only of 0’s. Hence, \( a = 0^n, \ b = 0^\beta, \ c = 0^{2^n-\alpha-\beta} \).

Then, by the pumping lemma with \( i = 2 \), \( ab^2 c = 0^n0^{2^\beta}0^{2^n-\alpha-\beta} = 0^{2^n+\beta} \in L_{exp} \).
But since \( 0 < |b| = \beta \leq n \) and \( \forall k \geq 0, \ k < 2^k \) then
\[ 2^n < 2^n + \beta \leq 2^n + n < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1} \]
So, \( 2^n + \beta \) is not a power of 2 since it is strictly between \( 2^n \) and \( 2^{n+1} \). But this means that \( ab^2 c \notin L_{exp} \) which is a contradiction. Thus, \( L_{exp} \) is not regular. 

\[ \blacksquare \]

3. \( L_{double} = \{ww \mid w \in \{0,1\}^*\} \)

Prove that \( L_{double} = \{ww \mid w \in \{0,1\}^*\} \) is not regular.

\textbf{Proof.} Suppose, for sake of contradiction, that \( L_{double} \) is regular.
By the pumping lemma, \( \exists \) a pumping length \( n \geq 0 \). Consider \( x = 0^n10^n1 \in L_{double} \).
By the pumping lemma, since \( |x| \geq n \), there exists strings \( a, b, c \) such that \( x = abc \), \( 0 < |b| \leq n \), \( |ab| \leq n \), and \( \forall i \geq 0, \ ab^i c \in L \). Since the first \( n \) bits of \( x \) consist only of 0’s, then \( a,b \) consist only of 0’s. Hence, \( a = 0^n, \ b = 0^\beta, \ c = 0^n-\alpha-\beta10^n1 \).
Then, by the pumping lemma with \( i = 2 \), \( ab^2 c = 0^n0^{2^\beta}0^{n-\alpha-\beta}10^n1 = 0^{n+\beta}10^n1 \in L_{double} \).
But since \( |b| = \beta > 0 \), then \( n + \beta > n \). But this means that \( ab^2 c \notin L_{double} \), which is a contradiction.
(To see that \( ab^2 c = 0^{n+\beta}10^n1 \notin L_{double} \), note that if \( ab^2 c = ww \) for some \( w \), then \( |w| = n + 1 + \beta/2 \) and each \( w \) must contain one of the two 1’s. So, \( \beta \) must be even. But if \( \beta > 1 \), then the second \( w \) contains both 1’s, which means the \( w \)’s are different.) Thus, \( L_{double} \) is not regular. 

\[ \blacksquare \]
4  \( L_\# \)

(1.52 in textbook)
Prove \( L_\# = \{ w \mid w = x_1\#x_2\#x_3\# \ldots \#x_k \text{ for } k \geq 0 \text{ each } x_i \in 1^*, \text{ and } x_i \neq x_j \text{ for } i \neq j \} \) is not regular.

**Proof.** Suppose, for sake of contradiction, that \( L_\# \) is regular.
By the pumping lemma, \( \exists \) a pumping length \( n \geq 0 \).
Consider \( x = 1^n\#1^{n+1}\#1^{n+2}\# \ldots \#1^{2n} \in L_\# \).
By the pumping lemma, since \( |x| \geq n \), there exists strings \( a, b, c \) such that \( x = abc \), \( 0 < |b| \leq n \), \( |ab| \leq n \), and \( \forall i \geq 0 \), \( ab^i c \in L_\# \). Since the first \( n \) bits of \( x \) consist only of 1’s, then \( a, b \) consist only of 1’s. Hence, \( a = 1^\alpha \), \( b = 1^\beta \), \( c = 1^{n-\alpha-\beta}\#1^{n+1}\#1^{n+2}\# \ldots \#1^{2n} \).
Then, by the pumping lemma with \( i = 2 \),
\[ ab^2c = 1^n12^n1^{n-\alpha-\beta}\#1^{n+1}\#1^{n+2}\# \ldots \#1^{2n} = 1^n+\beta\#1^{n+1}\#1^{n+2}\# \ldots \#1^{2n} \in L_\#. \]
But since \( 0 < |b| = \beta \leq 2n \), then \( n + 1 \leq n + \beta \leq 2n \). But this means that \( x_1 \) is equal to another \( x \) term. So, \( ab^2c \notin L_\# \), which is a contradiction. Thus, \( L_\# \) is not regular. 

\[ \square \]

5  \textbf{NOTPALIN}

(1.51c in textbook)
Prove that \( \text{NOTPALIN} = \{ w \mid w \in \{0, 1\}^* \text{ is not a palindrome} \} \) is not regular.

**Proof.** Suppose, for sake of contradiction, that \( \text{NOTPALIN} \) is regular.
Since \( \text{NOTPALIN} \) is regular, then its complement \( \text{PALIN} = \{ w \mid w \in \{0, 1\}^* \text{ is a palindrome} \} \) is regular. This is because the set of regular languages is closed under complement.
(i.e. If you had a DFA \( M \) that recognizes \( L \), then you can recognize its complement \( \overline{L} \) by making a new machine \( M' \) which is the same as \( M \) except that its accepting states \( F' = Q - F \) = all of the states that weren’t accepting states in \( M \).)
So, \( \text{PALIN} \) must be regular.
By the pumping lemma, \( \exists \) a pumping length \( n \geq 0 \). Consider \( x = 0^n1^n \in \text{PALIN} \).
By the pumping lemma, since \( |x| \geq n \), there exists strings \( a, b, c \) such that \( x = abc \), \( 0 < |b| \leq n \), \( |ab| \leq n \), and \( \forall i \geq 0 \), \( ab^i c \in \text{PALIN} \). Since the first \( n \) bits of \( x \) consist only of 0’s, then \( a, b \) consist only of 0’s. Hence, \( a = 0^\alpha \), \( b = 0^\beta \), \( c = 0^{n-\alpha-\beta}1^n \).
Then, by the pumping lemma with \( i = 2 \),
\[ ab^2c = 0^n0^2\beta0^{n-\alpha-\beta}1^n = 0^{n+\beta}1^n \in \text{PALIN} \).
But since \( |b| = \beta > 0 \), then \( ab^2c \notin \text{PALIN} \), which is a contradiction. Thus, \( \text{PALIN} \) is not regular, so \( \text{NOTPALIN} \) is not regular. 

\[ \square \]


Let $\Sigma = \{0, 1, +, =\}$.

Prove that $ADD = \{x=y+z \mid x, y, z \text{ are binary integers, and } x \text{ is the sum of } y \text{ and } z\}$ is not regular.

**Proof.** Suppose, for sake of contradiction, that $ADD$ is regular.

By the pumping lemma, $\exists$ a pumping length $n \geq 0$. Consider $w : 1^n=1^n+0 \in ADD$.

By the pumping lemma, since $|w| \geq n$, there exists strings $a, b, c$ such that $w = abc$, $0 < |b| \leq n$, $|ab| \leq n$, and $\forall i \geq 0$, $ab^i c \in ADD$. Since the first $n$ bits of $w$ consist only of 1’s, then $a, b$ consist only of 1’s. Hence, $a : 1^n$, $b : 1^\beta$, $c : 1^n=1^n+0$.

Then, by the pumping lemma with $i = 2$,

\[
ab^2 c : 1^n1^{2\beta}1^n=1^n+0 \in ADD.
\]

But since $|b| = \beta > 0$, then $1^n+\beta \neq 1^n+0$, so $ab^2 c \notin ADD$, which is a contradiction. Thus, $ADD$ is not regular.

\section{7 $L = \{01^k0^k \mid k \geq 0\}$}

Prove that $L = \{01^k0^k \mid k \geq 0\}$ is not regular.

**Proof.** Suppose, for sake of contradiction, that $L$ is regular.

By the pumping lemma, $\exists$ a pumping length $n \geq 0$. Consider $x = 01^n0^n \in L$.

By the pumping lemma, since $|x| \geq n$, there exists strings $a, b, c$ such that $x = abc$, $0 < |b| \leq n$, $|ab| \leq n$, and $\forall i \geq 0$, $ab^i c \in L$. Since the first $n$ bits of $x$ consist of a 0 followed by 1’s, then we have two cases:

1. Case 1: $b$ contains the first 0

   In this case, $a = \epsilon$, $b = 01^\beta$, $c = 1^n=0^n$.

   Then, by the pumping lemma with $i = 2$, $ab^2 c = 01^\beta01^\beta1^n=0^n \in L$.

   But since $ab^2 c$ contains a 0 between two 1’s, then $ab^2 c \notin L$, which is a contradiction.

2. Case 2: $b$ does not contain the first 0 (i.e. $b$ contains only 1’s) In this case, $a = 01^n$, $b = 1^\beta$, $c = 1^n=0^n$.

   Then, by the pumping lemma with $i = 2$, $ab^2 c = 01^n1^\beta1^n=0^n = 01^{n+\beta}0^n \in L$.

   But since $|b| = \beta > 0$, then $n + \beta > n$. Therefore, $ab^2 c \notin L$, which is a contradiction.

Since we reach a contradiction in both cases, then $L$ is not regular.