Math 31B Worksheet
Week 3

1. Suppose $a$ is a positive constant. Compute the limit

$$\lim_{x \to \infty} \frac{\ln(x)}{x^a}$$

What does this limit tell you about the growth of these functions? (You should answer using $\ll$ or $\gg$, and describing your answer in plain English.)

$$\ln x \ll x^a$$

2. (a) Repeat the previous problem, but this time comparing the functions $(\ln(x))^2$ and $x^a$. (Hint: You’ll have to do L’Hôpital’s Rule twice this time.)

$$\lim_{x \to \infty} \frac{(\ln(x))^2}{x^a} = \lim_{x \to \infty} \frac{2 \ln(x) \cdot \frac{1}{x}}{ax^{a-1}} = \lim_{x \to \infty} \frac{2 \ln(x)}{ax^a} = \infty$$

(b) Repeat the previous problem again, but this time comparing the functions $(\ln(x))^3$ and $x^a$. (Hint: You’ll have to do L’Hôpital’s Rule three times.)

$$\lim_{x \to \infty} \frac{(\ln(x))^3}{x^a} = \lim_{x \to \infty} \frac{3 \ln(x) \cdot \frac{1}{x}}{ax^{a-1}} = \lim_{x \to \infty} \frac{3 \ln(x)}{ax^a} = \infty$$

(c) Based on the work you did for the previous two parts, do you see a pattern? What would you conclude about the growth of $(\ln(x))^N$ (for any positive integer $N$) compared to $x^a$ (for any positive constant $a$)?

$$\ln \text{ general, } \lim_{x \to \infty} \frac{(\ln(x))^N}{x^a} = \ldots = \lim_{x \to \infty} \frac{N!}{a^N x^a} = 0 \text{ i.e. } (\ln(x))^N \ll x^a$$
(a) Compute \( \lim_{n \to \infty} \left( 4 - \frac{3}{n^4} \right) = 4 - 0 = 4 \)

(b) Use your answer to part (a) to compute \( \lim_{n \to \infty} \left( 4 - \frac{3}{n^4} \right)^{\frac{1}{2}} = \left( \lim_{n \to \infty} 4 - \frac{3}{n^4} \right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2 \)

What limit law are you using here? (You may want to discuss with your TA/LA, and/or look through the theorems in section 11.1 in your textbook.)

If \( f \) is continuous, then \( \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) \), i.e. we can move the limit inside the function.

3. Consider the sequence defined by the recurrence relation

\[ a_1 = 1, \quad a_n = 2a_{n-1} - \frac{3}{a_{n-1}} \]

Compute \( a_2, a_3, a_4, \) and \( a_5 \). (Feel free to use a calculator.)

\[
\begin{align*}
a_2 &= 2 \cdot 1 - \frac{3}{1} = 2 - 3 = -1 \\
a_3 &= 2(-1) - \frac{3}{(-1)} = -2 + 3 = 1 \\
a_4 &= 2 \cdot 1 - \frac{3}{1} = -1 \\
a_5 &= 2(-1) - \frac{3}{(-1)} = -2 + 3 = 1
\end{align*}
\]

*Note: The sequence can be rewritten as \( a_n = (-1)^{n-1} \). The sequence does not converge.