Midterm 1
Version A

Last Name: _______________________________
First Name: _______________________________
Last six digits of UID: _______________________

By signing below, you affirm that you have neither given nor received unauthorized help on this exam.

Signature: _______________________________

Instructions: Do not open this exam until instructed to do so. You will have 90 minutes to complete the exam. Please print your name and the last six digits of your student ID number above. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure your phone is silenced and stowed with your other belongings at the front of the room. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.

Please do not write below this line.

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1. (10 points) The thyroid hormones triiodothyronine ($T_3$) and thyroxine ($T_4$) regulate the metabolism of almost all tissues in the body. The thyroid is stimulated to produce these by a third hormone, thyrotropin, which is produced by the pituitary gland (so we will use the variable $P$ for the level of thyrotropin). Assume the following:

- Since thyrotropin ($P$) stimulates the production of $T_4$, the rate at which $T_4$ is produced is proportional to $P$, with a proportionality constant of 1.65.
- $T_4$ is converted to $T_3$ (mostly in the liver) at a per-mass rate of 23% per hour.
- The pituitary gland produces $P$ when both $T_3$ and $T_4$ are present. So assume that the rate of production of $P$ by the pituitary gland is proportional to the product of $T_3$ and $T_4$, with a proportionality constant of 0.03.
- $P$ is metabolized/excreted at a per-mass rate of 8% per hour.
- $T_3$ is metabolized/excreted at a per-mass rate of 11% per hour.

Write a system of differential equations for the levels of $T_3$, $T_4$, and $P$ in a person’s bloodstream. As usual, it is recommended that you start with a diagram.
Question 1 continued...  Last six digits of UID: ____________
2. (10 points) Ants and aphids have a well-documented mutualistic (symbiotic) relationship, as described below. Set up a differential equation model for the populations of ants \((N)\) and aphids \((P)\), based on the following assumptions:

- The aphids secrete a sugary substance called honeydew, which provides a food source for the ants. Therefore, the per-capita birth rate of the ants is proportional to the aphid population, with a proportionality constant of 0.03.
- 2\% of the ants die per day.
- The aphids have a per-capita birth rate of 12\% per day.
- Due to crowding, the per-capita death rate of the aphids is proportional to the aphid population, with proportionality constant 0.001.
- Aphids can also be killed by predators, but to protect their source of honeydew, the ants will aggressively defend the aphids: the more ants there are, the lower the risk of predation for the aphids. Thus, the per-capita predation rate for aphids is proportional to the inverse of the ant population, with a proportionality constant of 0.7.

As usual, it is recommended that you start with a diagram.
Question 2 continued... Last six digits of UID: ____________
3. (12 points) For each of the following, say whether or not the relation described is a function. If not, explain why not. (3 points each)

(a) \( X = \mathbb{R}_+, \quad Y = \mathbb{R}, \quad \text{and} \quad f: X \to Y \) defined by \( f(x) = \frac{x - 4}{x + 1} \).

(b) \( M = \{\text{musicians performing at Coachella}\}, \quad S = \{\text{songs}\}, \quad \text{and} \quad p: M \to S \) defined by

\[
p(m) = \text{songs to be performed by musician } m
\]

(c) \( X = \mathbb{R}, \quad Y = \mathbb{R}_+, \quad \text{and} \quad f: X \to Y \) defined by \( f(x) = \sqrt{x} \).

(d) \( S = \{\text{points on the surface of the Earth}\}, \quad E = \mathbb{R}, \quad \text{and} \quad \text{elev}: S \to E \) defined by

\[
\text{elev}(p) = \text{the elevation of point } p, \text{ in meters above sea level (negative if below sea level)}
\]
4. (8 points) The following graphs show the time series for a pair of state variables $L$ and $S$.

Sketch the corresponding trajectory on the axes provided below. *(Hint: It may help you to draw some vertical lines on the above graphs at appropriate $t$ values.)*
5. (10 points) Identify both a positive and a negative feedback loop in the following system. Justify your answer by drawing a feedback diagram.

\[
\begin{cases}
    A' = B \\
    B' = A + C \\
    C' = -A
\end{cases}
\]
6. (a) (5 points) On the axes below, sketch the vector field of the following system of differential equations:

\[ \begin{align*}
X' &= \frac{1}{2}X \\
Y' &= 0
\end{align*} \]

(Note: You may just plug in numbers, but you should be able to find an easy pattern and draw many many vectors quickly.)
(b) (5 points) On the axes below, sketch the vector field of the following system of differential equations:

\[
\begin{cases}
X' = 0 \\
Y' = -0.1Y^2
\end{cases}
\]

(Note: You may just plug in numbers, but you should be able to find an easy pattern and draw many many vectors quickly.)