1. Suppose that $a_n$ and $b_n$ are sequences of positive numbers, both of which have a limit of 0 as $n \to \infty$.

(a) If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$, then it means that $\frac{a_n}{b_n}$ goes to zero faster than $\frac{b_n}{b_n}$.

(b) If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$, then it means that $\frac{b_n}{b_n}$ goes to zero faster than $\frac{a_n}{a_n}$.

Note: If $\lim_{n \to \infty} \frac{a_n}{b_n}$ is neither 0 nor $\infty$, then it means that neither sequence goes to zero faster than the other: $a_n$ and $b_n$ are “equivalent” in terms of how fast they go to zero.

2. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) Suppose that $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$.

If $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} a_n$ converge.

If $\sum_{n=1}^{\infty} b_n$ diverge, then this test is inconclusive.

(b) Suppose that $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$.

If $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty} a_n$ diverge.

If $\sum_{n=1}^{\infty} b_n$ converge, then this test is inconclusive.

(c) Suppose that $\lim_{n \to \infty} \frac{a_n}{b_n}$ is neither 0 nor $\infty$. Then what is the relationship between the convergence/divergence of $\sum_{n=1}^{\infty} a_n$ and the convergence/divergence of $\sum_{n=1}^{\infty} b_n$?

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
3. Consider the series \( \sum_{n=3}^{\infty} \frac{4n - 2}{n(n - 1)(n + 1)} \). Call this \( \sum_{n=3}^{\infty} a_n \).

(a) What is the fastest growing term in the numerator? \( n^3 \)

If you were to multiply out the denominator, what would the fastest-growing term be? (You don’t need to actually multiply it out. Just think about what the largest exponent would be if you did.) \( n^3 \)

(b) Based on your answers to part (a), the terms in the series should go to zero as fast as (i.e., equivalent to) what?

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{p-series, } p=2 \quad \text{converges}
\]

(c) The thing you came up with in part (b)…let’s call it \( b_n \). So this defines a new series \( \sum_{n=1}^{\infty} b_n \). Does this new, much simpler series converge or diverge?

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \text{p-series, } p=2 \quad \text{converges}
\]

(d) Use the Limit Comparison Test to compare the original series \( \sum_{n=3}^{\infty} a_n \) to the one from part (c):

First, compute \( \lim_{n \to \infty} \frac{a_n}{b_n} \).

\[
\lim_{n \to \infty} \frac{\frac{4n}{n(n-1)(n+1)}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{\frac{4n}{n(n-1)(n+1)}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{4n^3}{1} = 4 \quad (4)
\]

Based on the value of that limit \( \text{and} \) whether or not \( \sum_{n=1}^{\infty} b_n \) converges/diverges,

\[
\sum_{n=1}^{\infty} \frac{4n - 2}{n(n - 1)(n + 1)} \quad \text{converges}
\]
4. Consider the series \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 - n + 5}} \). Call this \( \sum_{n=1}^{\infty} a_n \).

(a) In this case, the fastest-growing term in the denominator is \( n^3 \), but it’s also inside a square root.
So the whole denominator grows as fast as \( \frac{\sqrt{n^3}}{\sqrt{n^3 - n + 5}} \).
That means the terms in the series, \( \frac{n}{\sqrt{n^3 - n + 5}} \), go to zero like \( \frac{1}{n^{3/2}} \).

(b) Repeat the same logic as in the previous problem. First, based on your answer to part (a), what should the series \( \sum_{n=1}^{\infty} b_n \) be? Does it converge or diverge?
\[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ p-series, } p = \frac{3}{2} \text{ diverge} \]

(c) Use the Limit Comparison Test to compare the original series above to \( \sum_{n=1}^{\infty} b_n \):
First, compute \( \lim_{n \to \infty} \frac{a_n}{b_n} \).
\[
\lim_{n \to \infty} \frac{n}{\sqrt{n^3 - n + 5}} \frac{1}{\sqrt{n}} = \lim_{n \to \infty} \frac{n}{\sqrt{n^3 - n + 5}} \frac{1}{n^{3/2}} = \lim_{n \to \infty} \frac{1}{\sqrt{1 - \frac{n}{n^3}}}
\]
Based on the value of that limit and whether or not \( \sum_{n=1}^{\infty} b_n \) converges/diverges, does the original series above converge or diverge?
\[ \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 - n + 5}} \text{ diverge} \]

5. Consider the series \( \sum_{n=1}^{\infty} \frac{\ln(n + 1)}{n^{3/2}} \).
Remember that \( \ln(n) \) grows slower than any power of \( n \). So the numerator in this series grows slower than \( n^{0.000001} \). Therefore, you might think that you can try just comparing this to the series \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \).

(a) Based on what we just said (and on whether or not \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges), formulate a guess: do you think \( \sum_{n=1}^{\infty} \frac{\ln(n + 1)}{n^{3/2}} \) will converge, or diverge?
(b) Try using the Limit Comparison Test to compare \(\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}\) to \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\). What happens, and why?

\[
\lim_{n \to \infty} \frac{\ln(n+1)}{n^{3/2}} = \lim_{n \to \infty} \frac{\ln(n+1)}{n^{3/2}} = \infty \quad \text{but} \quad \sum \frac{1}{n^{3/2}} \text{ converges} \Rightarrow \text{inconclusive}
\]

(c) Okay, so \(\ln(n)\) grows slowly, but it's not completely insignificant in this problem. So let's try comparing our series to something that goes to zero a little slower than \(\frac{1}{n^{3/2}}\), like, say, \(\frac{1}{n}\). That is, try using the Limit Comparison Test to compare \(\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^{3/2}}\) to \(\sum_{n=1}^{\infty} \frac{1}{n}\). What is the problem this time?

\[
\lim_{n \to \infty} \frac{\ln(n+1)}{n^{3/2}} = \lim_{n \to \infty} \frac{\ln(n+1)}{n^{3/2}} = \frac{\ln(n+1)}{n+1} = \frac{2}{n+1} \Rightarrow 0
\]

but \(\sum \frac{1}{n}\) diverges \Rightarrow inconclusive

(d) Since neither attempt in (b) or (c) worked, what might you try now? Remember, you're trying to prove that the series \(\sum \frac{\ln(n+1)}{n^{3/2}}\) converges, so you need to compare to another series that does the same. But also you need to use an exponent in the denominator that's at least a little smaller than \(3/2\).

Come up with a \(b_n\) that satisfies those requirements, and try the Limit Comparison Test on it. You should be able to prove your guess from part (a).

We tried \(\sum \frac{1}{n^{1.5}}\) and \(\sum \frac{1}{n^{1}}\), try \(\sum \frac{1}{n^{1.25}}\) for some \(k \in (1, 1.5)\).

I will try \(p = 1.25\), \(\sum \frac{1}{n^{1.25}}\) converges since \(p = 1.25 > 1\)

\[
\lim_{n \to \infty} \frac{\ln(n+1)}{n^{1.25}} = \lim_{n \to \infty} \frac{\ln(n+1)}{n^{1.25}} = \frac{\ln(n+1)}{n^{1.25}} = \frac{\ln(n+1)}{n^{1.25}} = \frac{1}{n^{1.25}} = \frac{1}{n^{1.25}} = 0
\]

\(\Rightarrow \sum \frac{\ln(n+1)}{n^{3/2}}\) converges by 2(b)