1. (a) Define \( \lim_{n \to \infty} s_n = s \in \mathbb{R} \). (b) Prove that if \( \lim_{n \to \infty} s_n = s \in \mathbb{R} \), then the set \( S = \{ s_n : n \in \mathbb{N} \} \) is bounded.

\[
\text{(5 points) (a) Given } \epsilon > 0 \text{ there exists } N \text{ such that } n > N \implies |s_n - s| < \epsilon.
\]

\[
\text{(15 points) (b) Given } \epsilon = 1 \text{ there exists } N \in \mathbb{N} \text{ such that } n > N \implies |s_n - s| < 1 \text{ so }
\]

\[
|s_n| = |s_n - s + s| \\
\leq |s_n - s| + |s| < 1 + |s|
\]

Therefore

\[
|s_n| \leq \max \{ |s|, \ldots, |s_N|, 1 + |s| \}
\]

for all \( n \in \mathbb{N} \).
2. (a) Define \( \lim_{n \to \infty} s_n = -\infty \). (b) Prove that if \( \lim_{n \to \infty} s_n = -\infty \) and \( \lim_{n \to \infty} t_n = t \in \mathbb{R} \), then \( \lim_{n \to \infty} (s_n + t_n) = -\infty \).

(5 points) (a) Given \( m < 0 \) there exists \( N \) such that \( n > N \) implies \( s_n < m \).

(15 points) (b) Since \( \{t_n : n \in \mathbb{N}\} \) is bounded (Problem 1), there exists \( B > 0 \) such that \( |t_n| < B \) for all \( n \in \mathbb{N} \) and thus \( -B < t_n < B \). Given \( m < 0 \) there exists \( N \) such that \( n > N \) implies \( s_n < m - B \). So \( s_n + t_n < (m - B) + B = m \) and therefore \( \lim_{n \to \infty} s_n + t_n = -\infty \).
3. Let $S$ be a bounded subset of $\mathbb{R}$. (a) Define the infimum $\inf(S)$ of $S$. (b) Let $-S = \{ -s : s \in S \}$. Prove that $\inf(S) = -\sup(-S)$.

(5 points) (a) $s_0 = \inf(S)$ if $s_0 \leq s$ for all $s \in S$ and if $t$ is a lower bound for $S$ then $t \leq s_0$.

(15 points) (b) Since $S$ is bounded, so is $-S$. Let $s_0 = \sup(-S)$ so $s_0 \geq -s$ for all $s \in S$ and thus $-s_0 \leq s$. Let $t$ be a lower bound for $S$ so $t \leq s$ for all $s \in S$. Then $-t \geq -s$ and since $-t$ is an upper bound for $-S$ then $-t \geq s_0$ and therefore $t \leq -s_0 = -\sup(-S)$ and $-\sup(-S) = \inf(S)$. 
4. (a) Define \( \liminf s_n \). (b) Prove that if \( \lim_{n \to \infty} s_n = s \in \mathbb{R} \), then \( \liminf s_n \geq s \) by proving that \( \liminf s_n \geq s - \epsilon \) for all \( \epsilon > 0 \).

(5 points) (a) \( S_N = \{ s_n : n > N \} \), \( u_N = \inf(S_N) \)

\[ \lim \inf s_n = \lim_{N \to \infty} u_N \]

(15 points) (b) Given \( \epsilon > 0 \) there exists \( N \) such that \( n > N \) implies \( |s_n - s| < \epsilon \) and therefore \(-\epsilon < s_n - s < \epsilon\) and \( s - \epsilon < s_n \) and \( s - \epsilon \) is a lower bound for \( S_N \). Then \( \inf(S_N) \geq s - \epsilon \). Since the sequence \( (u_N) \) is increasing

\[ \lim \inf s_n = \lim_{N \to \infty} u_N \geq s - \epsilon \]
5. Let \( S \subseteq \mathbb{R} \) be a nonempty bounded subset such that \( s_0 = \sup(S) \notin S \). Prove that there exists an increasing sequence \((s_n) \subseteq S\) such that \( \lim_{n \to \infty} s_n = s_0 \).

By the definition of supremum, there exists \( s_1 \in S \) such that \( s_0 - 1 < s_1 < s_0 \).

Assume that there exist \( s_1 < s_2 < \ldots < s_n \), all in \( S \), such that \( s_0 - \frac{1}{n} < s_j < s_0 \) for \( j = 1, \ldots, n \). There exists \( s_{n+1} \in S \) such that

\[
\max \{ s_0 - \frac{1}{n+1}, s_n \} < s_{n+1} < s_0.
\]

Therefore \((s_n)\) exists by induction and \( \lim_{n \to \infty} s_n = s_0 \) by the Squeeze Theorem.