1. (10 points) The assumptions below describe a very basic model of a viral infection in vertebrate animals. This model considers only the number of viruses \( (V) \), the number of cells infected by the virus \( (I) \), and the number of antibodies \( (A) \) produced by the body to fight the infection.

- When a cell is infected, the virus begins producing copies of itself within the cell, which are released when the cell dies. So the rate of production of new viruses is proportional to the number of infected cells, with proportionality constant \( m \).
- The per-capita death rate of infected cells is approximately 50% per hour.
- Free viruses don't last long on their own, so the per-capita death rate of the viruses is 200% per hour.
- The immune system produces antibodies at a rate that is proportional to the number of viruses in the body, with a proportionality constant of \( p \).
- Antibodies act to prevent the infection of cells, so the more antibodies there are, the lower the infection rate. Of course, the more viruses there are, the higher the infection rate. So assume that the infection rate (rate at which the number of infected cells increases) is proportional to the number of viruses, with proportionality constant \( r \), times the inverse of the number of antibodies.
- Finally, antibodies leave the system at a per-capita rate of 4% per hour.

Write down a system of differential equations for this model. As usual, it is recommended that you start with a diagram.

\[
\begin{align*}
   V' &= mI - 2V \\
   I' &= r \frac{V}{A} - 0.5I \\
   A' &= pV - 0.04A
\end{align*}
\]

Question 1 continues on the next page...
2. The body mass (in kg) of a fish of age \( t \) (in months) is given by a function

\[
M(t) = \frac{7t}{10 + t}
\]

Use this to answer each of the following questions about the growth rate of the fish.

(a) (3 points) What is the average growth rate of the fish between the ages of 8 months and 12 months?

\[
\text{Average rate} = \frac{M(12) - M(8)}{12 - 8} = \frac{\frac{84}{22} - \frac{56}{18}}{4} = \frac{\frac{35}{198}}{4} = 0.1767
\]

(b) (4 points) Use \( \Delta t = 0.01 \) to estimate the instantaneous growth rate of the fish at age \( t = 8 \) months.

\[
\frac{M(8 + \Delta t) - M(8)}{\Delta t} = \frac{M(8.01) - M(8)}{0.01} = \frac{3.113 - 3.111}{0.01} = 0.2159
\]

(c) (4 points) Calculate the exact value of the instantaneous growth rate of the fish at age \( t = 8 \) months.

\[
M'(t) = \frac{(7) \cdot (10 + t) - (7t) \cdot (1)}{(10 + t)^2} = \frac{70 + 7t - 7t}{(10 + t)^2} = \frac{70}{(10 + t)^2}
\]

\[
M'(8) = \frac{70}{(10 + 8)^2} = \frac{70}{324} = 0.2160
\]
3. (a) (6 points) Find the equation of the line tangent to the graph of 
\( f(x) = \sqrt[3]{x} \) at \( x = 64 \).

\[
\begin{align*}
f(x) &= x^{\frac{1}{3}} \\
f'(x) &= \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{x^{\frac{2}{3}}} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}
\end{align*}
\]

Slope of tangent line: \( f'(64) = \frac{1}{3 \cdot \sqrt[3]{64^2}} = \frac{1}{3 \cdot 4^2} = \frac{1}{48} \)

Point on tangent line: \((x_1, y_1) = (64, f(64)) = (64, 4)\)

Eq. for tangent line: 
\[
y - y_1 = m(x - x_1) \\
y - 4 = \frac{1}{48} (x - 64)
\]

\[
y = 4 + \frac{1}{48} (x - 64) \quad \text{or} \quad y = \frac{3}{4} x + \frac{8}{3}
\]

(b) (4 points) Use your answer to part (a) to approximate the value of \( \sqrt[3]{60} \).

\[
\sqrt[3]{60} = f(60) \approx 4 + \frac{1}{48} (60 - 64) \\
= 4 + \frac{1}{48} (-4) \\
= 4 - \frac{1}{12} \\
= \frac{47}{12} = 3.916666
\]

Actual value: \( \sqrt[3]{60} = 3.9148676\ldots \)

(Difference is 0.001799\ldots, which is 0.046\%.)
4. From chemistry, you may remember that the ideal gas law states that for \( n \) moles of an ideal gas,

\[
P V = n R T,
\]

where \( R = 0.082 \), and \( P, V, \) and \( T \) are the pressure (in atm, atmospheres), volume (in L, liters), and temperature (in K, Kelvin), respectively. Suppose you have 1 mole of an ideal gas at 300 K, so that its volume is

\[
V = \frac{1 \cdot 0.082 \cdot 300}{P} = \frac{24.6}{P}.
\]

(a) (5 points) Suppose the current pressure is \( P = 2 \) atm. Write down the linear approximation formula (either form) for the volume at this pressure.

\[
\text{Short form: } \Delta V \approx \left( \frac{dV}{dP} \right)_{P=2} \Delta P
\]

\[
V = \frac{24.6}{P}, \quad \text{so} \quad \frac{dV}{dP} = -\frac{24.6}{P^2}
\]

\[
\frac{dV}{dP} \bigg|_{P=2} = -\frac{24.6}{2^2} = -6.15
\]

So \( \Delta V \approx -6.15 \Delta P \) when \( \Delta P \approx 0 \)

(Or the long form: \( V \approx 12.3 + (-6.15)(P-2) \) when \( P \approx 2 \))

(b) (3 points) Use your linear approximation to estimate how much the volume of the gas will change if the pressure increases by 0.2 atm.

(Note: You may check that your answer is in the right ballpark by just plugging numbers into the equation above, but to get credit for this, you must use a linear approximation to arrive at your answer.)

\[
\text{Given } \Delta P = +0.2 \ \text{atm} \quad \text{Find } \Delta V \quad \text{(approx)}
\]

\[
\Delta V \approx -6.15 \cdot 0.2 = -1.23
\]

So the volume should decrease by approximately \( 1.23 \text{ L} \).
5. The three parts of this problem are unrelated to each other.

(a) (4 points) Let \( f(t) = t^4 - 5t^3 - 3t + 9 \). Find the instantaneous rate of change of \( f \) at \( t = 2 \).

\[
f'(t) = 4t^3 - 15t^2 - 3
\]

\[
f'(2) = 4 \cdot 2^3 - 15 \cdot 2^2 - 3
\]

\[
= 4 \cdot 8 - 15 \cdot 4 - 3
\]

\[
= -31
\]

(b) (4 points) Let \( g(t) = \sin(t^3 - t) \). Find the slope of the tangent line to the graph of \( g \) at \( t = 1 \).

\[
g'(t) = \cos(t^3 - t) \cdot (3t^2 - 1)
\]

\[
g'(1) = \cos(1^3 - 1) \cdot (3 \cdot 1^2 - 1)
\]

\[
= \cos(0) \cdot (2) = 1 \cdot 2 = 2
\]

(c) (5 points) Let \( h(t) = (3t - 7)e^t \). Starting at \( t = 0 \), if we increase \( t \) by 0.03, then approximately how much will \( h(t) \) change by?

\[
h'(t) = (3) \cdot e^t + (3t - 7) \cdot e^t
\]

\[
= e^t \cdot (3 + 3t - 7) = e^t \cdot (3t - 4)
\]

\[
h'(0) = e^0 \cdot (3 \cdot 0 - 4) = 1 \cdot (-4) = -4
\]

Linear approximation: \( \Delta h \approx h'(0) \cdot \Delta t \)

\[
\Delta h \approx -4 \cdot 0.03 = -0.12
\]

In other words, \( h \) will decrease by approx. 0.12.
4. (10 points) The following system of differential equations describes the concentrations of two hormones, gonadotropin-releasing hormone \((G)\) and testosterone \((T)\), in the bloodstream of an adult male.

\[
\begin{align*}
G' &= \frac{1}{1+T} - G \\
T' &= G - 0.1T
\end{align*}
\]

Suppose that at time \(t = 0\), the concentrations of these hormones are \(G = 2.5\) and \(T = 1\). Use Euler's method, with a step size of \(\Delta t = 0.1\), to approximate the values of \(G\) and \(T\) at time \(t = 0.3\).

| \(t\) | Current State \([\begin{array}{c} G \\ T \end{array}]\) | Change Vector \([\begin{array}{c} G' \\ T' \end{array}]\) | Next State \([\begin{array}{c} G \\ T \end{array}]\) \\
<table>
<thead>
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<tr>
<td>0</td>
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<td>([\begin{array}{c} -2 \ 2.4 \end{array}])</td>
<td>([\begin{array}{c} 2.5 \ 1 \end{array}] + 0.1 \times \begin{bmatrix} -2 \ 2.4 \end{bmatrix} = \begin{bmatrix} 2.3 \ 1.24 \end{bmatrix})</td>
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<td>([\begin{array}{c} 2.3 \ 1.24 \end{array}])</td>
<td>([\begin{array}{c} -1.854... \ 2.176 \end{array}])</td>
<td>([\begin{array}{c} 2.3 \ 1.24 \end{array}] + 0.1 \times \begin{bmatrix} -1.854... \ 2.176 \end{bmatrix} = \begin{bmatrix} 2.115... \ 1.958... \end{bmatrix})</td>
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<td>0.2</td>
<td>([\begin{array}{c} 2.115... \ 1.458... \end{array}])</td>
<td>([\begin{array}{c} -1.708... \ 1.969... \end{array}])</td>
<td>([\begin{array}{c} 2.115... \ 1.458... \end{array}] + 0.1 \times \begin{bmatrix} -1.708... \ 1.969... \end{bmatrix} = \begin{bmatrix} 1.944... \ 1.654... \end{bmatrix})</td>
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<tr>
<td>0.3</td>
<td>([\begin{array}{c} 1.944... \ 1.654... \end{array}])</td>
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\[\text{Scratch work:}\]
\[G = 2.5, \; T = 1: \; G' = \frac{1}{1+1} - 2.5 = -2, \quad T' = 2.5 - 0.1(1) = 2.4\]
\[G = 2.3, \; T = 1.24: \; G' = \frac{1}{1+1.24} - 2.3 = -1.854...\]
\[T' = 2.3 - 0.1(1.24) = 2.176\]
\[G = 2.115, \; T = 1.458: \; G' = \frac{1}{1+1.458} - 2.115 = -1.708...\]
\[T' = 2.115 - 0.1(1.458) = 1.969...\]

\[\text{At} \; t = 0.3, \; G \approx 1.944, \; T \approx 1.654\]