Further Exercise 1.4.3
On a hot day, students are lining up to buy ice cream. Let $L$ be the number of people in line. Write a differential equation for $L$ using the following assumptions.
- Students join the line at a rate proportional to the number of people already in line, with a proportionality constant of 0.1.
- Students get ice cream and leave the line at a constant rate of 0.4 per minute.
- Students get tired of standing in line and leave at a per capita rate proportional to the number of people in line, with a proportionality constant of 0.02.

Solution

$$L' = 0.1L - 0.4 - 0.02L^2$$
$$= -0.02L^2 + 0.1L - 0.4$$

Further Exercise 1.4.11
In this textbook, we will use capital letters for state variables and lowercase letters for parameters, but many models in the scientific literature don’t follow this convention. Both state variables and parameters can be written as either uppercase or lowercase letters. In the examples below, identify the state variables and parameters. (Hint: Think about what state variables do that parameters don’t, or see the footnote on page 26.)

a) $g' = 0.2g$

b) $a' = 0.35ab$, $b' = 2b$

c) $X' = aX + RW$, $W' = RX$

d) $c' = Qcd - Rd$, $d' = PdRc$

Solution

key: Only the one that changes (have prime) is state variable

a) State variables: $g$

b) State variables: $a, b$

c) State variables: $X, W$  Parameters: $a, R$

d) State variables: $c, d$  Parameters: $Q, R, P$
**Exercise 2.2.1**
A bowling lane is 60 feet long. If a bowling ball is released at $t = 0$ and reaches the pins 2.5 seconds later, what is its average speed?

**Solution**
Average speed $= \frac{60 \text{ (ft.)}}{2.5 \text{ (sec.)}} = 24 \frac{\text{ft.}}{\text{sec.}}$

**Exercise 2.2.2**
Let's say we are given the functional form of the curve in Figure 2.2:

$$f(t) = (B - A) \frac{t^4}{1 + t^4} + A$$

Here we assume $A = 0$ and $B = 1$. Calculate estimates of the average speed over several intervals beginning at $t = 1$, say $\Delta t = 0.5$, 0.2, and 0.1.

**Solution**
For $A=0$, $B=1$, $f(t)$ can be rewritten as:

$$f(t) = \frac{t^4}{1 + t^4} = 1 - \frac{1}{1 + t^4}$$

$$f(t + \Delta t) - f(t) = 1 - \frac{1}{1 + (t + \Delta t)^4} - 1 + \frac{1}{1 + t^4}$$

$$= \frac{1}{1 + t^4} - \frac{1}{1 + (t + \Delta t)^4}$$

Since $t = 1$

$$\frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{\frac{1}{2} - \frac{1}{1 + (1 + \Delta t)^4}}{\Delta t}$$

$\Delta t = 0.5$

$$\frac{\frac{1}{2} - \frac{1}{1 + (1+0.5)^4}}{0.5} = 0.6701$$

$\Delta t = 0.2$

$$\frac{\frac{1}{2} - \frac{1}{1 + (1+0.2)^4}}{0.2} = 0.87324$$
\[
\triangle t = 0.1 \\
\frac{\frac{1}{2} - \frac{1}{1 + (1+0.1)^2}}{0.1} = 0.94172
\]

**Exercise 2.2.3**
If at some instant an object's speed is 30 \( \text{mi} \) \( h^{-1} \), will it travel 30 miles in the next hour?

**Solution**
No, the instantaneous speed only means the speed at that specific moment, which cannot be applied on the next hour.

**Exercise 2.2.4**
Notice that our calculation results in a negative number. Why does this make sense?

**Solution**
It is reasonable since the slope of the function with respect to time is negative.

**Exercise 2.2.5**
Approximate \( H'(1.5) \) using the time interval \( \triangle t = 0.0001 \).

**Solution**
The original \( H(t) \) function is

\[
H(t) = 1000 - 16t^2 \\
H'(1.5) = \frac{-16(1.5 + 0.0001)^2 + 16 \times 1.5^2}{0.0001} = -48.0016
\]

**Exercise 2.2.6**
Why do we not allow \( \triangle t \) to reach 0?

**Solution**
First, first derivative means instantaneous "rate of change", which must involves some "change" with respect to the original time \( t \). Having \( \triangle t = 0 \) typically means "no change". Mathematically, the definition of first derivative is \( \lim_{\triangle t \to 0} \frac{f(t+\triangle t) - f(t)}{\triangle t} \), by
having $\triangle t = 0$ will be $\frac{0}{0}$. This result is not only not defined (divided by zero) but not related to $f(t)$.

**Exercise 2.2.7**
Use successive approximations to find the objects speed at $t = 1$ second.

**Solution**
*You may choose any $\triangle t$*

\[ H'(t = 1) \bigg|_{\triangle t} = \frac{-16(1 + \triangle t)^2 + 16}{\triangle t} \]

\[
H(1) \bigg|_{\triangle t = 0.0001} = -32.0016 \\
H(1) \bigg|_{\triangle t = 0.000001} = -32.00016 \\
H(1) \bigg|_{\triangle t = 0.0000001} = -32.000016
\]

We can conclude that $H'(1)$ approximates to -32.

**Exercise 2.2.8**
Carry out a similar calculation for $t = 2$.

**Solution**

\[ H'(t = 1) \bigg|_{\triangle t} = \frac{-16(2 + \triangle t)^2 + 16 \times 4}{\triangle t} \]

\[
H(1) \bigg|_{\triangle t = 0.0001} = -64.016 \\
H(1) \bigg|_{\triangle t = 0.000001} = -64.0016 \\
H(1) \bigg|_{\triangle t = 0.0000001} = -64.00016
\]

We can conclude that $H'(1)$ approximates to -64.
Exercise 2.2.9
Use this result to find the objects velocity at t = 2.

Solution

\[-32 \times 2 = -64\text{ft/sec}\]

Exercise 2.2.10
The function describing how air pressure varies with elevation is

\[P(H) = 101352e^{-\frac{0.28H}{2396}}\]

where P is measured in pascals and H in meters. Approximate the rate of change of P with respect to H at a height of 2000 meters.

Solution
Let \(\Delta H = 0.00001\)

\[
\frac{P(2000 + 0.00001) - P(2000)}{0.00001} = -9.37563
\]

Exercise 2.3.1
Calculate the slope of the secant line to the graph of \(Y = \frac{X}{1+X}\) from \(X = 1\) to \(X = 3\).

Solution
The slope of a secant line of two points can be define as:

\[
\frac{f(x_a) - f(x_b)}{x_a - x_b}
\]

or

\[
\frac{f(x_b) - f(x_a)}{x_b - x_a}
\]

We define \(x_a = 3, x_b = 1\), and use the first approach:

\[
Y = \frac{3}{1+3} - \frac{1}{1+1} = \frac{3}{4} - \frac{1}{2} = \frac{1}{2}
\]

\[
\frac{1}{2}
\]
Exercise 2.3.2
Find the slope of the secant line crossing the graph of \( f(t) = 200 - 16t^2 \) at the following values of \( t \). What value is the slope approaching?

a) \( t=2, t=2.5 \)  b) \( t=2, t=2.1 \)  c) \( t=2, t=2.05 \)

Solution

\[
\text{Slope} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}
\]

a) Slope = \( \frac{f(2.5) - f(2)}{(2.5-2)} = 16 \times \frac{-2.5^2 + 2^2}{0.5} = -72 \)

b) Slope = \( \frac{f(2.1) - f(2)}{(2.1-2)} = 16 \times \frac{-2.1^2 + 2^2}{0.1} = -65.6 \)

c) Slope = \( \frac{f(2.05) - f(2)}{(2.05-2)} = 16 \times \frac{-2.05^2 + 2^2}{0.05} = -64.8 \)

The slope is approaching -64. (It is ok to say it approach 64.1 or 64.2, etc, since the above calculation is not sufficient).

Exercise 2.3.3
Find the equation of the tangent line to \( f(t) = 200 - 16t^2 \) at \( t = 2 \).

Solution

From Exercise 2.3.2, the slope of tangent line at \( t = 2 \) is -64.

The equation is:

\[
y = m(t - 2) + f(2) \\
= -64 \times (t - 2) + 136 \\
= -64t + 264
\]

Exercise 2.3.4
Write equations for the following lines in both slope intercept and pointslope form.

a) The line that has a slope of 2 and a Y intercept of -54.

b) The line that has a slope of -3 and passes through the point (2, 6).

c) The line that passes through the points (1, 7) and (3, 5).

Solution
a) \[ Y = 2X - 54 \]

b) \[ Y = -3(X - 2) + 6 \]
\[ = -3X + 12 \]

c) \[ Y = \frac{5 - 7}{3 - 1}(X - 1) + 7 \]
\[ = -X + 1 + 7 \]
\[ = -X + 8 \]

Exercise 2.4.3
What is the complete equation for the tangent line to \( Y = f(X) \) at the point \((X_0, f(X_0))\)?

Solution
\[ Y = \frac{dY}{dX}\bigg|_{X_0} \times (X - X_0) + f(X_0) \]

Exercise 2.4.4
In SageMath, pick a function and a point on the function. Plot the function at several magnification levels. Describe what you see.

Solution
The function we use here is \( f(x) = 20\sin(52x) + x^3 \), expand at \( x = 5 \)
As we zoom in, the slope became more linear.
Exercise 2.4.5

In the example of the falling object, we calculated its velocity $H'(1.5)$, the rate of change of height with respect to time, at 1.5 seconds after it was released. We got the answer $-48 \text{ ft/s}$. Now estimate how far the ball will drop in the next 0.01 seconds. In other words, let $\Delta t = 0.01$ seconds, and calculate an approximate value for $\Delta H$.

Solution

$$\Delta H = H'(1.5) \times \Delta t = -48 \frac{\text{ft}}{\text{s}} \times 0.01 \text{s} = -0.48 \text{ft}$$

Exercise 2.4.6

The equation for the height of the falling ball is:

$$H(t) = H(0) - 16t^2$$

Use this equation to calculate the actual change in $H$ from $t = 1.5 \text{ s}$ to $t = 1.51 \text{ s}$. How close is this actual $\Delta H$ to the $\Delta H$ you calculated in Exercise 2.4.5?
Solution

$$\triangle H = H(1.51) - H(1.5)$$

$$= 16 \times (-1.51^2 + 1.5^2)$$

$$= -0.4816 \text{ ft}$$

The result we get from Exercise 2.4.5 is close to the real value (error = \(\frac{-0.4816 + 0.48}{-0.4816} = -0.33\%\) since \(\triangle t\) is small.

Exercise 2.4.7

View \(f(X) = |X|\) at several zoom levels and show that the corner at \(X = 0\) remains a sharp corner no matter how closely you zoom in. Briefly explain why this means that it does not have a derivative at \(X = 0\).

Solution

If we approach from the right hand side, for \(\triangle X > 0, \triangle X \to 0\)

$$\lim_{\triangle X \to 0, \triangle X > 0, X = 0} \frac{f(X + \triangle X) - f(X)}{\triangle X} = \frac{\triangle X}{\triangle X} = 1$$

If we approach from the left hand side, for \(\triangle X < 0, \triangle X \to 0\)

$$\lim_{\triangle X \to 0, \triangle X < 0, X = 0} \frac{f(X + \triangle X) - f(X)}{\triangle X} = \frac{-\triangle X}{\triangle X} = -1$$

It is obvious that the slope change dramatically at \(X = 0\) (not continuous), thus, \(f(X)\) does not have a derivative at \(X = 0\).

Exercise 2.4.8

Is the function in Figure 2.15 differentiable at all points shown other than \(X = 0\)?

Solution

Yes, from Exercise 2.4.7 we know

$$f(X) = |X|, f'(X)|_{X < 0} = -1, f'(X)|_{X > 0} = 1$$

All derivatives are continuous except \(X = 0\).

For \(f(X) = \sqrt{|X|}, f(X) = f(-X)\) (symmetric), if we can prove one side we
can show the other side also works.

For \( X > 0 \), \( f(X) = \sqrt{X} \) the derivative of \( \sqrt{X} \) is:

\[
f'(X) = \lim_{\Delta X \to 0} \frac{\sqrt{X + \Delta X} - \sqrt{X}}{\Delta X} \\
= \lim_{\Delta X \to 0} \frac{\sqrt{X + \Delta X} - \sqrt{X}}{\Delta X} \times \frac{\sqrt{X + \Delta X} + \sqrt{X}}{\sqrt{X + \Delta X} + \sqrt{X}} \\
= \lim_{\Delta X \to 0} \frac{\Delta X \times (\sqrt{X + \Delta X} + \sqrt{X})}{\Delta X} \\
= \lim_{\Delta X \to 0} \frac{1}{\sqrt{X + \Delta X} + \sqrt{X}} \\
\therefore \Delta X \to 0 \\
= \frac{1}{2\sqrt{X}}
\]

We can see \( f'(X) \) exist for all \( X > 0 \), since it is symmetric, and \( \frac{1}{2\sqrt{X}} \) is continuous for all \( X > 0 \), we proved that \( f'(X) \) exists for all \( X \neq 0 \).