Definition 1 (Def of PP:counting1). An arrival process \((T_k)_{k \geq 1}\) is said to be a Poisson process of rate \(\lambda > 0\) if its associated counting process \((N(t))_{t \geq 0}\) satisfies the following properties:

(i) \(N(0) = 0\);
(ii) (Independent increment) For any \(t, s \geq 0\), \(N(t + s) - N(t)\) is independent of \((N(u))_{u \leq t}\);
(iii) For any \(t, s \geq 0\), \(N(t + s) - N(t)\) is Poisson(\(\lambda s\)).

Definition 2 (Def of PP:counting2). A counting process \((N(t))_{t \geq 0}\) is said to be a Poisson process with rate \(\lambda > 0\) if it satisfies the following conditions:

(i) \(N(0) = 0\);
(ii) \(P(N(t) = 0) = 1 - \lambda t + o(t)\);
(iii) \(P(N(t) = 1) = \lambda t + o(t)\);
(iv) \(P(N(t) = 2) = o(t)\);
(v) (Independent increment) For any \(t, s \geq 0\), \(N(t + s) - N(t)\) is independent of \((N(u))_{u \leq t}\);
(vi) (Stationary increment) For any \(t, s \geq 0\), the distribution of \(N(t + s) - N(t)\) is invariant under \(t\).

Exercise 3. Let \((N(t))_{t \geq 0}\) be a counting process with the properties (i)-(iii) in Def 1. Let \(T_k = \inf\{u \geq 0 | N(u) = k\}\) be the \(k\)th arrival time and let \(\tau_k = T_k - T_{k-1}\) be the \(k\)th inter-arrival time.

(i) Use conditioning on \(T_k\) to show that for any \(k \geq 1\) and \(s \geq 0\), \(N(T_k + s) - N(T_k)\) is independent of \((N(u))_{u \leq T_k}\).

(ii) Let \(Z(t) = \inf\{u \geq 0 | N(t + u) > N(t)\}\) be the waiting time for the first arrival after time \(t\). Show that \(Z(t) \sim \text{Exp}(\lambda)\) for all \(t \geq 0\).

(iii) Use (ii) and conditioning on \(T_{k-1}\) to show that \(\tau_k \sim \text{Exp}(\lambda)\) for all \(k \geq 1\).

Exercise 4. Let \((N(t))_{t \geq 0}\) is the Poisson process with rate \(\lambda > 0\) in the sense of Definition 2. Denote \(f_n(t) = P(N(t) = n)\) for each \(n \geq 0\).

(i) Show that
\[ P(N(t) \leq n - 2, N(t + h) = n) \leq P(N(t + h) - N(t) \geq 2). \] (1)

Conclude that
\[ P(N(t) \leq n - 2, N(t + h) = n) = o(h). \] (2)

(ii) Use (i) and independent/stationary increment properties to show that
\[ f_n(t + h) = P(N(t + h) = n) = P(N(t) = n, N(t + h) - N(t) = 0) + P(N(t) = n - 1, N(t + h) - N(t) = 1) + P(N(t) \leq n - 2, N(t + h) = n) \]
\[ = f_n(t)(1 - \lambda h + o(h)) + f_{n-1}(t)(\lambda h + o(h)) + o(h). \] (6)

(iii) Use (ii) to show that the following differential equation holds:
\[ \frac{d f_n(t)}{dt} = -\lambda f_n(t) + \lambda f_{n-1}(t). \] (7)

(iv) By multiplying the integrating factor \(\mu(t) = e^{\lambda t}\) to (7), show that
\[ (e^{\lambda t} f_n(t))' = \lambda e^{\lambda t} f_{n-1}(t). \] (8)

Use the initial condition \(f_n(0) = P(N(0) = n) = 0\) to derive the recursive equation
\[ f_n(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda s} f_{n-1}(s) \, ds. \] (9)
(v) Use induction to conclude that \( f_n(t) = (\lambda t)^n e^{-\lambda t} / n! \).
(vi) Conclude that for all \( t, s \geq 0 \) and \( n \geq 0 \),
\[
N(t + s) - N(s) \sim \text{Poisson}(\lambda t).
\] (10)

**Definition 5** (Nonhomogeneous PP). An arrival process \((T_k)_{k \geq 1}\) is said to be a Poisson process with rate \( \lambda(t) \) if its counting process \((N_t)_{t \geq 0}\) satisfies the following properties:

(i) \( N(0) = 0 \).
(ii) \((N(t))_{t \geq 0}\) has independent increments.
(iii) For any \( 0 \leq s < t \), \( N(t) - N(s) \sim \text{Poisson}(\mu) \) where
\[
\mu = \int_s^t \lambda(r) \, dr.
\] (11)

**Exercise 6.** Let \((T_k)_{k \geq 1} \sim \text{PP}(\lambda(t))\). Let \((\tau_k)_{k \geq 1}\) be the inter-arrival times.

(i) Let \( Z(t) \) be the waiting time for the first arrival after time \( t \). Show that
\[
P(Z(t) \geq x) = \exp \left( - \int_t^{t+x} \lambda(t) \, dt \right). \] (12)

(ii) From (i), deduce that \( \tau_1 \) has PDF
\[
f_{\tau_1}(t) = \lambda(t) e^{-\int_0^t \lambda(r) \, dr}.
\] (13)

(iii) Denote \( \mu(t) = \int_0^t \lambda(s) \, ds \). Use (i) and conditioning to show
\[
P(\tau_2 > x) = E_{\tau_1}[P(\tau_2 > x | \tau_1)] |_{\tau_1} = \int_0^\infty P(\tau_2 > x | \tau_1 = t) f_{\tau_1}(t) \, dt \] (14)
\[
= \int_0^\infty e^{-\mu(t+x)-\mu(t)} \lambda(t) e^{-\mu(t)} \, dt \] (15)
\[
= \int_0^\infty \lambda(t) e^{-\mu(t+x)} \, dt.
\] (16)

Conclude that \( \tau_1 \) and \( \tau_2 \) does not necessarily have the same distribution.

**Exercise 7.** Let \((N_0(t))_{t \geq 0}\) be the counting process of a Poisson process of rate 1. Let \( \lambda(t) \) denote a non-negative function of \( t \), and let \( m(t) = \int_0^t \lambda(s) \, ds \).

Define \( N(t) \) by
\[
N(t) = N_0(m(t)) = \# \text{ arrivals during } [0, m(t)].
\] (19)

Show that \((N(t))_{t \geq 0}\) is the counting process of a Poisson process of rate \( \lambda(t) \).