Exercise 1 (Reward from Markov process). Let \((X_k)_{k \geq 0}\) be an irreducible and aperiodic Markov chain on state space \(\Omega = \{1, 2, \cdots, m\}\) with transition matrix \(P = (p_{ij})\). Let \(\pi\) be the unique stationary distribution of the chain.

Suppose the chain spends an independent amount of time at each state \(x \in \Omega\), whose distribution \(F_x\) may depend only on \(x\). For each real \(t \geq 0\), let \(Y(t) \in \Omega\) denote the state of the chain at time \(t\). (This is a continuous-time Markov process.)

(i) Fix \(x \in \Omega\), and let \(\tilde{T}^{(x)}_k\) denote the \(k\)th time that the Markov process \((Y(t))_{t \geq 0}\) returns to \(x\). Let \((\tilde{T}^{(x)}_k)_{k \geq 1}\) and \((N^{(x)}(t))_{t \geq 0}\) be the associated inter-arrival times and the counting process, respectively. Then

\[ N^{(x)}(t) = \text{number of visits to } x \text{ that } (Y(t))_{t \geq 0} \text{ makes up to time } t. \]

Show that \((\tilde{T}^{(x)}_k)_{k \geq 1}\) is a renewal process. Moreover, show that

\[ \mathbb{P}\left( \lim_{n \to \infty} \frac{N^{(x)}(t)}{t} = \frac{1}{\mathbb{E}[\tilde{T}^{(x)}_1]} \right) = 1. \]

(ii) Let \(T_k\) denote the \(k\)th time that the Markov process \((Y(t))_{t \geq 0}\) jumps. Let \((\tau_k)_{k \geq 1}\) and \((N(t))_{t \geq 0}\) be the associated inter-arrival times and the counting process, respectively. Show that

\[ N(t) = N^{(1)}(t) + N^{(2)}(t) + \cdots + N^{(m)}(t). \]

Use (i) to derive that

\[ \mathbb{P}\left( \lim_{n \to \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[\tilde{T}^{(1)}]} \right) = 1. \]

(iii) Let \(g : [0, \infty) \to \mathbb{R}\) be a reward function and fix \(x \in \Omega\). Define

\[ R^{(x)}(t) = \sum_{k=1}^{N(t)} g(\tau_k) \mathbf{1}(X_k = x). \]

Namely, every time the Markov process \((Y(t))_{t \geq 0}\) visits \(x\) and spends \(\tau_k\) amount of time, we get a reward of \(g(\tau_k)\). Show that as \(t \to \infty\),

\[ \lim_{n \to \infty} \frac{R^{(x)}(t)}{t} = \left( \frac{1}{\mathbb{E}[\tilde{T}^{(1)}]} + \cdots + \frac{1}{\mathbb{E}[\tilde{T}^{(m)}]} \right) \mathbb{E}[g(\tau_k) \mid X_k = x] \pi(x) \quad \text{a.s.} \]

You may use the fact that (see Exercise 5.13 in Lecture note 2)

\[ \mathbb{P}\left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(X_k = k) = \pi(x) \right) = 1. \]

Exercise 2 (Alternating renewal process). Let \((\tau_k)_{k \geq 1}\) be a sequence of independent RVs where

\[ \mathbb{E}[\tau_{2k-1}] = \mu_1, \quad \mathbb{E}[\tau_{2k}] = \mu_2 \quad \forall k \geq 1. \]

Define and arrival process \((T_k)_{k \geq 1}\) by \(T_k = \tau_1 + \cdots + \tau_k\) for all \(k \geq 1\).

(i) Is \((T_k)_{k \geq 1}\) a renewal process?

(ii) Let \((X_k)_{k \geq 0}\) be a Markov chain on state space \(\Omega = \{1, 2\}\) with transition matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Show that the chain has \(\pi = [1/2, 1/2]\) as the unique stationary distribution.
(iii) Suppose the chain spends time \( \tau_k \) at state \( X_k \in \Omega \) in between the \( k-1 \)st and \( k \)th jump. For each real \( t \geq 0 \), let \( Y(t) \in \Omega \) denote the state of the chain at time \( t \). Let \( N(t) \) be the number of jumps that \( Y(t) \) makes up to time \( t \). Define

\[
R^{(1)}(t) = \sum_{k=1}^{N^{(1)}(t)} \tau_k 1(X_k = 1),
\]

which is the total amount of time that \( (Y(t))_{t \geq 0} \) spends at state 1. Use Exercise 1 to deduce

\[
P\left( \lim_{n \to \infty} \frac{N^{(1)}(t)}{t} = \frac{\mu_1}{\mu_1 + \mu_2} \right) = 1.
\]

Exercise 3 (Poisson janitor). A light bulb has a random lifespan with distribution \( F \) and mean \( \mu_F \). A janitor comes at times according to \( \text{PP}(\lambda) \) and checks and replaces the bulb if it is burnt out. Suppose all bulbs have independent lifespans with the same distribution \( F \).

(i) Let \( T_k \) be the \( k \)th time that the janitor arrives and replaces the bulb. Show that \( (T_k)_{k \geq 0} \) with \( T_0 = 0 \) is a renewal process.

(ii) Let \( (\tau_k)_{k \geq 1} \) be the inter-arrival times of the renewal process defined in (i). Using the memoryless property of Poisson processes to show that

\[
E[\tau_k] = \mu_F + 1/\lambda \quad \forall k \geq 1.
\]

(iii) Let \( N(t) \) be the number of bulbs replaced up to time \( t \). Show that

\[
P\left( \lim_{n \to \infty} \frac{N(t)}{t} = \frac{1}{\mu_F + 1/\lambda} \right) = 1.
\]

(iv) Let \( B(t) \) be the total duration that bulb is working up to time \( t \), that is,

\[
B(t) = \int_0^t 1(\text{Bulb is on at time } s) \, ds.
\]

Use renewal reward process to show that

\[
P\left( \lim_{n \to \infty} \frac{B(t)}{t} = \frac{\mu_F}{\mu_F + 1/\lambda} \right) = 1.
\]

(v) Let \( V(t) \) denote the total number of visits that the janitor has made by time \( t \). Show that

\[
P\left( \lim_{n \to \infty} \frac{N(t)}{V(t)} = \frac{1/\lambda}{\mu_F + 1/\lambda} \right) = 1.
\]

That is, the fraction of times that the janitor replaces the bulb converges to \( \frac{1/\lambda}{\mu_F + 1/\lambda} \) almost surely, which is also the fraction of times that the bulb is off by (iv).