Exercise 1 (Sum of i.i.d. Exp is Erlang). Let $X_1, X_2, \ldots, X_n \sim \text{Exp}(\lambda)$ be independent exponential RVs.

(i) Show that $f_{X_1+X_2}(z) = \lambda^2 ze^{-\lambda z} 1(z \geq 0)$.

(ii) Show that $f_{X_1+X_2+X_3}(z) = 2^{-1} \lambda^3 z^2 e^{-\lambda z} 1(z \geq 0)$.

(iii) Let $S_n = X_1 + X_2 + \cdots + X_n$. Use induction to show that $S_n \sim \text{Erlang}(n, \lambda)$, that is,

$$f_{S_n}(z) = \frac{\lambda^n z^{n-1} e^{-\lambda z}}{(n-1)!}. \quad (1)$$

Sol. I will only show (iii) since this implies the previous parts. We use induction on $n$. There is nothing to show for the base case. Let $S_k = X_k + X_{k+1}$ and $S_k, X_{k+1}$ are independent. Hence the PDF of $S_{k+1}$ is given by the convolution of the PDFs of $S_k$ and $X_{k+1}$. Hence we get

$$f_{S_{k+1}}(z) = \int_0^z \frac{\lambda^k}{(k-1)!} e^{-\lambda t} t^{k-1} \lambda e^{-\lambda (z-t)} \, dt
= \frac{\lambda^{k+1}}{(k-1)!} \int_0^z t^{k-1} e^{-\lambda z} \, dt
= \frac{\lambda^{k+1}}{(k-1)!} e^{-\lambda z} \frac{z^k}{k} \quad (4)$$

as desired. \qed

Exercise 2 (Memoryless property of exponential RV). A continuous positive RV $X$ is said to have memoryless property if

$$P(X \geq t_1 + t_2) = P(X \geq t_1)P(X \geq t_2) \quad \forall x_1, x_2 \geq 0. \quad (5)$$

(i) Show that (5) is equivalent to

$$P(X \geq t_1 + t_2 | X \geq t_2) = P(X \geq t_1) \quad \forall x_1, x_2 \geq 0. \quad (6)$$

(ii) Show that exponential RVs have memoryless property.

(iii) Suppose $X$ is continuous, positive, and memoryless. Let $g(t) = \log P(X \geq t)$. Show that $g$ is continuous at 0 and

$$g(x + y) = g(x) + g(y) \quad \text{for all } x, y \geq 0. \quad (7)$$

Using Exercise 3, conclude that $X$ must be an exponential RV.

Sol. (i) Easy.

(ii) First part is easy. Note that $g(t)$ is continuous at $t = 0$ since $g(0) = \log P(X \geq 0) = \log 1 = 0$ and

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \log P(X \geq x) = \log \left( \lim_{x \to 0} P(X \geq x) \right) = \log 1 = 0, \quad (8)$$

where we used continuity of log and CDF. Thus by the following exercise $g(t) = ct$ for some constant $c$. This yields

$$P(X \geq t) = e^{g(t)} = e^{ct}. \quad (9)$$

This yields $X$ is an exponential RV. \qed
**Exercise 3.** Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function with the property that $g(x + y) = g(x) + g(y)$ for all $x, y \geq 0$. Further assume that $g$ is continuous at 0. In this exercise, we will show that $g(x) = cx$ for some constant $c$.

(i) Show that $g(0) = g(0 + 0) = g(0) + g(0)$. Deduce that $g(0) = 0$.

(ii) Show that for all integers $n \geq 1$, $g(n) = ng(1)$.

(iii) Show that for all integers $n, m \geq 1$,

$$ng(1) = g(n \cdot 1) = g(m(n/m)) = mg(n/m).$$

Deduce that for all nonnegative rational numbers $r$, we have $g(r) = rg(1)$.

(iv) Show that $g$ is continuous.

(v) Let $x$ be nonnegative real number. Let $r_k$ be a sequence of rational numbers such that $r_k \to x$ as $k \to \infty$. By using (iii) and (iv), show that

$$g(x) = g \left( \lim_{k \to \infty} r_k \right) = \lim_{k \to \infty} g(r_k) = g(1) \lim_{k \to \infty} r_k = x \cdot g(1).$$

**Proof.** Follow the steps. For (iv), since $g$ is continuous at 0, note that

$$\lim_{h \to 0} g(x + h) = \lim_{h \to 0} (g(x) + g(h)) = g(x) + \lim_{h \to 0} g(h) = g(x) + g(0) = g(x + 0) = g(x).$$

\[\square\]

**Exercise 4** (Sum of independent Poisson RVs is Poisson). Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent Poisson RVs. Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

**Sol.** Omitted. \[\square\]