Exercise 1 (Reward from Markov process). Let \((X_k)_{k \geq 0}\) be an irreducible and aperiodic Markov chain on state space \(\Omega = \{1, 2, \cdots, m\}\) with transition matrix \(P = (p_{ij})\). Let \(\pi\) be the unique stationary distribution of the chain.

Suppose the chain spends an independent amount of time at each state \(x \in \Omega\), whose distribution \(F_x\) may depend only on \(x\). For each real \(t \geq 0\), let \(Y(t) \in \Omega\) denote the state of the chain at time \(t\). (This is a continuous-time Markov process.)

(i) Fix \(x \in \Omega\), and let \(T_{x}^{(k)}\) denote the \(k\)th time that the Markov process \((Y(t))_{t \geq 0}\) returns to \(x\). Let \((\tau_{k}^{(x)})_{k \geq 1}\) and \((N^{(x)}(t))_{t \geq 0}\) be the associated inter-arrival times and the counting process, respectively. Then
\[
N^{(x)}(t) = \text{number of visits to } x \text{ that } (Y(t))_{t \geq 0} \text{ makes up to time } t. \tag{1}
\]

Show that \((T_{x}^{(k)})_{k \geq 1}\) is a renewal process. Moreover, show that
\[
\mathbb{P}\left( \lim_{n \to \infty} \frac{N^{(x)}(t)}{t} = \frac{1}{\mathbb{E}[\tilde{\tau}_{1}]} \right) = 1. \tag{2}
\]

(ii) Let \(T_{x}\) denote the \(k\)th time that the Markov process \((Y(t))_{t \geq 0}\) jumps. Let \((\tau_{k})_{k \geq 1}\) and \((N(t))_{t \geq 0}\) be the associated inter-arrival times and the counting process, respectively. Show that
\[
N(t) = N^{(1)}(t) + N^{(2)}(t) + \cdots + N^{(m)}(t). \tag{3}
\]

Use (i) to derive that
\[
\mathbb{P}\left( \lim_{n \to \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[\tilde{\tau}_{1}]} + \cdots + \frac{1}{\mathbb{E}[\tilde{\tau}_{m}]} \right) = 1. \tag{4}
\]

(iii) Using the fact that (see Exercise 5.13 in Lecture note 2)
\[
\mathbb{P}\left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1(X_{k} = k) = \pi(x) \right) = 1, \tag{5}
\]
show that
\[
\mathbb{P}\left( \lim_{t \to \infty} \frac{N^{(x)}(t)}{N(t)} = \pi(x) \right) = 1. \tag{6}
\]

(iv) Let \(g : [0, \infty) \to \mathbb{R}\) be a reward function and fix \(x \in \Omega\). Use strong law of large numbers to show
\[
\lim_{t \to \infty} \frac{1}{N^{(x)}(t)} \sum_{k=1}^{N(t)} g(\tau_{k}) 1(X_{k} = x) = \mathbb{E}[g(\tau_{k}) | X_{k} = x] \quad \text{a.s.} \tag{7}
\]

(v) Define
\[
R^{(x)}(t) = \sum_{k=1}^{N(t)} g(\tau_{k}) 1(X_{k} = x) \tag{8}
\]
Namely, every time the Markov process \((Y(t))_{t \geq 0}\) visits \(x\) and spends \(\tau_{k}\) amount of time, we get a reward of \(g(\tau_{k})\). Writing
\[
\frac{R^{(x)}(t)}{t} = \frac{N(t)}{t} \frac{N^{(x)}(t)}{N(t)} \frac{1}{N^{(x)}(t)} \sum_{k=1}^{N(t)} g(\tau_{k}) 1(X_{k} = x), \tag{9}
\]
show that as \(t \to \infty\),
\[
\lim_{n \to \infty} \frac{R^{(x)}(t)}{t} = \left( \frac{1}{\mathbb{E}[\tilde{\tau}_{1}]} + \cdots + \frac{1}{\mathbb{E}[\tilde{\tau}_{m}]} \right) \pi(x) \mathbb{E}[g(\tau_{k}) | X_{k} = x] \quad \text{a.s.} \tag{10}
\]
Exercise 2 (Alternating renewal process). Let \((\tau_k)_{k \geq 1}\) be a sequence of independent RVs where

\[
E[\tau_{2k-1}] = \mu_1, \quad E[\tau_{2k}] = \mu_2 \quad \forall k \geq 1.
\] (11)

Define and arrival process \((T_k)_{k \geq 1}\) by \(T_k = \tau_1 + \cdots + \tau_k\) for all \(k \geq 1\).

(i) Is \((T_k)_{k \geq 1}\) a renewal process?

(ii) Let \((X_k)_{k \geq 0}\) be a Markov chain on state space \(\Omega = \{1, 2\}\) with transition matrix

\[
P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\] (12)

Suppose \(X_0 = 0\). Show that the chain has \(\pi = [1/2, 1/2]\) as the unique stationary distribution.

(iii) Suppose the chain spends time \(\tau_k\) at state \(X_k \in \Omega\) in between the \(k-1\)st and \(k\)th jump. For each real \(t \geq 0\), let \(Y(t) \in \Omega\) denote the state of the chain at time \(t\). Let \(N(t)\) be the number of jumps that \(Y(t)\) makes up to time \(t\). Define

\[
R^{(1)}(t) = \sum_{k=1}^{N^{(1)}(t)} \tau_k 1(X_k = 1),
\] (13)

which is the total amount of time that \((Y(t))_{t \geq 0}\) spends at state 1. Use Exercise 1 to deduce

\[
P\left( \lim_{n \to \infty} \frac{R^{(1)}(t)}{t} = \frac{\mu_1}{\mu_1 + \mu_2} \right) = 1. \] (14)

Exercise 3 (Poisson janitor). A light bulb has a random lifespan with distribution \(F\) and mean \(\mu_F\). A janitor comes at times according to \(\text{PP}(\lambda)\) and checks and replace the bulb if it is burnt out. Suppose all bulbs have independent lifespans with the same distribution \(F\).

(i) Let \(T_k\) be the \(k\)th time that the janitor arrives and replaces the bulb. Show that \((T_k)_{k \geq 0}\) with \(T_0 = 0\) is a renewal process.

(ii) Let \((\tau_k)_{k \geq 1}\) be the inter-arrival times of the renewal process defined in (i). Using the memoryless property of Poisson processes to show that

\[
E[\tau_k] = \mu_F + 1/\lambda \quad \forall k \geq 1.
\] (15)

(iii) Let \(N(t)\) be the number of bulbs replaced up to time \(t\). Show that

\[
P\left( \lim_{n \to \infty} \frac{N(t)}{t} = \frac{1}{\mu_F + 1/\lambda} \right) = 1. \] (16)

(iv) Let \(B(t)\) be the total duration that bulb is working up to time \(t\), that is,

\[
B(t) = \int_0^t 1(\text{Bulb is on at time } s) \, ds.
\] (17)

Use renewal reward process to show that

\[
P\left( \lim_{n \to \infty} \frac{B(t)}{t} = \frac{\mu_F}{\mu_F + 1/\lambda} \right) = 1. \] (18)

(v) Let \(V(t)\) denote the total number of visits that the janitor has made by time \(t\). Show that

\[
P\left( \lim_{n \to \infty} \frac{N(t)}{V(t)} = \frac{1/\lambda}{\mu_F + 1/\lambda} \right) = 1. \] (19)

That is, the fraction of times that the janitor replaces the bulb converges to \(\frac{1/\lambda}{\mu_F + 1/\lambda}\) almost surely, which is also the fraction of times that the bulb is off by (iv).