1. Let \( f(x) = \arctan(x) \).

(a) (8 points) Compute \( T_3(x) \), the Taylor polynomial of degree 3 for \( f \), centered at 0.

\[
\begin{align*}
  f(x) &= \arctan(x) \\
  f'(x) &= \frac{1}{1+x^2} = (1+x^2)^{-1} \\
  f''(x) &= -(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2} \\
  f'''(x) &= \frac{(-2)(1+x^2)x - (-2x)\cdot 2(1+x^2)x \cdot 2x}{(1+x^2)^3} \\
  &= \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} = \frac{6x^2-2}{(1+x^2)^3} \\

  f(0) &= \arctan(0) = 0 \\
  f'(0) &= \frac{1}{1+0} = 1 \\
  f''(0) &= \frac{0}{(1+0)^3} = 0 \\
  f'''(0) &= \frac{0-2}{(1+0)^3} = -2 \,
\end{align*}
\]

\[
T_3(x) = 0 + 1 \cdot (x - 0) + \frac{0}{2!} (x-0)^2 + \frac{-2}{3!} (x-0)^3 \\
= x - \frac{1}{3} x^3
\]

(b) (2 points) Use your answer to part (a) to approximate \( \arctan(1) \).

\[
\arctan(1) \approx T_3(1) = 1 - \frac{1}{3} = \frac{2}{3}
\]

(Note: As you may know, \( \tan(\frac{\pi}{4}) = 1 \), so \( \arctan(1) = \frac{\pi}{4} \approx 0.785 \), so this is not a very good approximation.)
2. (8 points) Assume $f$ is a (one-to-one) function and $g$ is the inverse of $f$. The table to the right gives the values of $f(x)$ and $f'(x)$ for several values of $x$. Use this to compute $g'(-3)$. You must justify your answer!

By the Inverse Function Theorem,

$$g'(x) = \frac{1}{f'(g(x))}$$

So

$$g'(-3) = \frac{1}{f'(g(-3))}$$

Since $f'(0) = -3$, $g(-3) = 0$, so

$$f'(g(-3)) = f'(0) = -4.$$ 

So

$$g'(-3) = \frac{1}{f'(g(-3))} = \frac{1}{f'(0)} = \frac{1}{-4} = -\frac{1}{4}$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>9</td>
<td>-1</td>
</tr>
<tr>
<td>-4</td>
<td>7</td>
<td>-2</td>
</tr>
<tr>
<td>-3</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>-5</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-8</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>-12</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>-17</td>
<td>-6</td>
</tr>
<tr>
<td>5</td>
<td>-24</td>
<td>-7</td>
</tr>
</tbody>
</table>
3. Use the Limit Comparison Test or Direct Comparison Test to determine whether each of the following series converges or diverges.

(a) (5 points) \( \sum_{n=1}^{\infty} \frac{n^2 \ln(n)}{n^3 - 3n^2 + 7} \)

**Intuition:** Fastest-growing terms on top and bottom are \( \frac{n^2 \cdot \ln(n)}{n^3} \) which is slightly *bigger* than \( \frac{1}{n} \) and \( \sum \frac{1}{n} \) diverges. So this should diverge. Let's compare to \( \frac{1}{n} \).

**LCT:** (using \( \leq b_n = \leq \frac{1}{n} \))

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2 \cdot \ln(n)}{n^3 - 3n^2 + 7} = \lim_{n \to \infty} \frac{n^3 \cdot \ln(n)}{n^3 - 3n^2 + 7} \cdot \frac{1}{n^3}
\]

\[= \lim_{n \to \infty} \frac{\ln(n)}{1 - \frac{3}{n} + \frac{7}{n^3}} = \infty \]

Since \( L = \infty \) and \( \leq b_n \) diverges, \( \leq a_n \) diverges also.

---

Question 3 continues on the next page...
(b) (6 points) \[ \sum_{n=1}^{\infty} \frac{n + 5}{\sqrt{n^5 + n^2 - 1}} \]

Intuition: Fastest-growing terms on top and bottom are
\[ \frac{n}{n^{2.5}} \quad \Rightarrow \quad \frac{1}{n^{1.5}} \]

So let's compare to \( \frac{1}{n^{1.5}} \).

**LCT:** (using \( \sum b_n = \sum \frac{1}{n^{1.5}} \))

\[ L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + 5}{\sqrt{n^5 + n^2 - 1}} \times \frac{\frac{1}{n^{1.5}}}{\frac{1}{n^{1.5}}} = \lim_{n \to \infty} \frac{(n + 5)}{\sqrt{n^5 + n^2 - 1}} \]

\[ = \lim_{n \to \infty} \frac{n^{2.5} + 5n^{1.5}}{n^{5/2} + n - 1} \]

\[ = \lim_{n \to \infty} \frac{n^{2.5} + 5n^{1.5}}{n^{5/2} (n^{5/2} + n^{1/2} - 1)} \cdot \frac{\frac{1}{n^{2.5}}}{\frac{1}{n^{2.5}}} \]

\[ = \lim_{n \to \infty} \frac{1 + 5 \frac{1}{n}}{\sqrt{1 + \frac{5}{n^{1/2}}} - 1} = \frac{1}{\sqrt{1}} = 1 \]

Since \( 0 < L < \infty \) and \( \sum b_n = \sum \frac{1}{n^{1.5}} \) converges (because \( p = 1.5 > 1 \)), then \( \sum a_n \) converges also.
4. For each of the following series, show (using any method you wish) that it converges absolutely, converges conditionally, or diverges.

(a) (6 points) \( \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{\sqrt{n}} \)

Check absolute convergence: \( \sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln(n)}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}} \)

\( \frac{\ln(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} \) (for all \( n \geq 3 \)) and \( \leq \frac{1}{\sqrt{n}} \)

diverges, so by DCT, \( \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}} \) diverges also.

So the series does \textbf{not} converge \textbf{absolutely}.

\textbf{Alternating Series Test:}

- \( \lim_{n \to \infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n \to \infty} \frac{1}{\frac{1}{2} n^{-1/2}} = \lim_{n \to \infty} \frac{2\sqrt{n}}{n} = \lim_{n \to \infty} \frac{2}{\sqrt{n}} = 0 \) \[ \text{by L'Hopital's Rule} \]

- \( \frac{\ln(n)}{\sqrt{n}} \) is decreasing: Its derivative is \( \frac{x\sqrt{n} - \ln(n)x^n^{-\frac{3}{2}}}{n^{3/2}} \)

\( = \frac{1 - \frac{3}{2} \ln(n)}{n} < 0 \) (for \( n > e^2 \))

Therefore the series converges, by \textbf{Alt. Series Test}. Since it converges, but not \textbf{absolutely}, it \textbf{converges conditionally}.

(b) (5 points) \( \sum_{n=1}^{\infty} \frac{(-5)^n}{n^n} \)

\textbf{Root test:} \( R = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\frac{(-5)^n}{n^n}} \)

\( = \lim_{n \to \infty} \sqrt[n]{\frac{5^n}{n^n}} = \lim_{n \to \infty} \frac{5}{n} = 0, \)

Since \( R < 1 \), the series \textbf{converges absolutely}, by the \textbf{Root Test}.

Question 4 continues on the next page...
(c) (5 points) \( \sum_{n=1}^{\infty} \frac{(-1)^n n!}{2^n} \)

One way: Ratio Test:

\[
R = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1)!}{2^{n+1}} \cdot \frac{2^n}{(-1)^n \cdot n!} \right| = \lim_{n \to \infty} \frac{(n+1)!}{2^n} \cdot \frac{2^n}{n!} = \lim_{n \to \infty} \frac{n+1}{2} = \infty
\]

Since \( R > 1 \), the series diverges, by the Ratio Test.

Another way: Just use the Divergence Test, if you can show that \( \lim_{n \to \infty} \frac{n!}{2^n} \neq 0 \).

In fact, \( n! \gg 2^n \), so this means \( \lim_{n \to \infty} \frac{n!}{2^n} = \infty \), so \( \lim_{n \to \infty} \frac{(-1)^n n!}{2^n} \neq 0 \), so by the Divergence Test, the series diverges.