Further Exercises 3.2 - 2

(a) The equilibrium points are determined to be $P = 0$, $P = 1/3$, and $P = 1$. Using the linear stability analysis with

$$\frac{dP'}{dP} = \frac{d}{dP} (6P^3 - 8P^2 + 2P) = 18P^2 - 16P + 2,$$

we can determine the stability of each one:

$$\left.\frac{dP'}{dP}\right|_{P=0} = 2 > 0 \Rightarrow \text{unstable};$$
$$\left.\frac{dP'}{dP}\right|_{P=1/3} = -4/3 < 0 \Rightarrow \text{stable};$$
$$\left.\frac{dP'}{dP}\right|_{P=1} = 4 > 0 \Rightarrow \text{unstable}.$$

(b) Since $P = 0.1$ is between the unstable equilibrium 0 and the stable equilibrium 1/3, 33.3% of the population will have the new gene in the long run.

(c) Since $P = 0.9$ is between the stable equilibrium 1/3 and the unstable equilibrium 1, 33.3% of the population will have the new gene in the long run.

Further Exercises 3.2 - 3

The equilibrium point is determined to be $L = k$. Using the linear stability analysis with

$$\frac{dL'}{dL} = -r,$$

we can determine its stability:

$$\left.\frac{dL'}{dL}\right|_{L=k} = -r < 0 \Rightarrow \text{stable}.$$

Thus, the organism will eventually grow to $k$. 
Further Exercises 3.2 - 4
The equilibrium points are determined to be \( X = 0 \) and \( X = X(0)e^{k/\alpha} \). Using the linear stability analysis with
\[
\frac{dX'}{dX} = k - \alpha \ln \left( \frac{X}{X(0)} \right) + X \cdot \left( -\alpha \cdot \frac{X(\theta)}{X} \cdot \frac{1}{X(\theta)} \right) = k - \alpha - \alpha \ln \left( \frac{X}{X(0)} \right),
\]
we can determine the stability of each one:
\[
\left. \frac{dX'}{dX} \right|_{X=0} = +\infty > 0 \Rightarrow \text{unstable};
\left. \frac{dX'}{dX} \right|_{X=X(0)e^{k/\alpha}} = -\alpha < 0 \Rightarrow \text{stable}.
\]
Thus, the tumor will eventually grow to \( X(0)e^{k/\alpha} \).

Further Exercises 3.2 - 7
No. It’s impossible to draw the situation.

Exercise 3.3.1
The equilibrium point is determined to be \((X^*, Y^*) = (0, 0)\).

Exercise 3.3.2
The change vector is determined to be \((X', Y') = (3, -4)\).

Exercise 3.3.3
Figure 1 shows time series for the system \( X' = X, Y' = -Y \) with initial condition \((X(0), Y(0)) = (1, 2)\).

![Figure 1: Time series for the system \( X' = X, Y' = -Y \).](image.png)
Exercise 3.4.1
We can derive
\[
\begin{align*}
S' &= 0.01ST - 0.2S = 0 \\
T' &= 0.05T - 0.01ST = 0
\end{align*}
\Rightarrow \begin{cases} S = 0 \text{ or } T = 20 \\ T = 0 \text{ or } S = 5 \end{cases}
\]
and thus the equilibrium points \((S^*, T^*)\) are \((0, 0)\) and \((5, 20)\).

Exercise 3.4.2
We can set the \(M'\) differential equation to zero:
\[
M' = M(2 - M - 0.5D) = 0 \Rightarrow M = 0 \text{ or } M = 2 - 0.5D
\]
and therefore
\[
M\text{-nullclines} \begin{cases} M = 0 \\ M = 2 - 0.5D \end{cases}
\]

Exercise 3.4.3
Each intersection point of the nullclines satisfies \(D' = M' = 0\), which is exactly the equilibrium point. No, they cannot.

Exercise 3.4.4
We can set the \(M'\) differential equation to zero:
\[
M' = M(2 - D - M) = 0 \Rightarrow M = 0 \text{ or } M = 2 - D
\]
and therefore
\[
M\text{-nullclines} \begin{cases} M = 0 \\ M = 2 - D \end{cases}
\]

Exercise 3.4.5
We can derive
\[
\begin{align*}
D' &= 3D - 2MD - D^2 = 0 \\
M' &= 2M - DM - M^2 = 0
\end{align*}
\Rightarrow \begin{cases} D = 0 \text{ or } M = -D/2 + 3/2 \\ M = 0 \text{ or } M = 2 - D \end{cases}
\]
and thus the equilibrium points \((D^*, M^*)\) are \((0, 0)\), \((0, 2)\), \((3, 0)\), and \((1, 1)\).
Exercise 3.4.6
We can pick the following test points for the $M$-nullclines:

- test point 1: $(D', M')_{(1,0)} = (2, 0)$ change vector points right
- test point 2: $(D', M')_{(4,0)} = (-4, 0)$ change vector points left
- test point 3: $(D', M')_{(0.5,1.5)} = (-0.25, 0)$ change vector points left
- test point 4: $(D', M')_{(1.5,0.5)} = (0.75, 0)$ change vector points right

You can check your results with Figure 3.27 in the textbook.

Exercise 3.4.7
Note that the left equilibrium point is a low $D$/high $M$ state. The right equilibrium point is the opposite, a high $D$/low $M$ state. Thus, the saddle point is a switch between the two behaviors.

Exercise 3.4.8
The $N$-nullclines are determined to be
\[ N' = N(0.05 - 0.01P) = 0 \Rightarrow N = 0 \text{ or } P = 5 \]
and the $P$-nullclines are determined to be
\[ P' = P(0.005N - 0.1) = 0 \Rightarrow P = 0 \text{ or } N = 20. \]
Thus the equilibrium points $(N^*, P^*)$ are $(0, 0)$ and $(20, 5)$. Figure 2 shows the vector field of the Lotka-Volterra predation model.

![Figure 2: Vector field of the Lotka-Volterra predation model.](image-url)
Exercise 3.4.9
We can set the \( M' \) differential equation to zero:

\[
M' = M(r_M - k_M D - c_M M) = 0 \Rightarrow M = 0 \text{ or } M = \frac{k_M}{c_M} D + \frac{r_M}{c_M}
\]

and therefore

\[
M\text{-nullclines } \begin{cases} 
M = 0 \\
M = \frac{k_M}{c_M} D + \frac{r_M}{c_M}
\end{cases}
\]

Further Exercises 3.4 - 1

(a) Suppose we have an equilibrium point \((N^*, P^*)\), where \(N^* > 0\) and \(P^* = 0\). We can plug them into the differential equation of \(N\) and get \(N' = rN^* \neq 0\). Therefore, \((N^*, P^*)\) cannot be an equilibrium point.

Similarly, suppose we have another equilibrium point \((N^*, P^*)\), where \(N^* = 0\) and \(P^* > 0\). We can plug them into the differential equation of \(P\) and get \(P' = -\delta P^* \neq 0\). Therefore, \((N^*, P^*)\) cannot be an equilibrium point.

(b) We can derive

\[
\begin{cases} 
N' = rN - aNP = 0 \\
P' = caNP - \delta P = 0
\end{cases} \Rightarrow \begin{cases} 
N = 0 \text{ or } P = r/a \\
P = 0 \text{ or } N = \delta / ca
\end{cases}
\]

and thus the equilibrium points \((N^*, P^*)\) are \((0, 0)\) and \((\delta / ca, r/a)\).

Further Exercises 3.4 - 2
We can derive

\[
\begin{cases} 
N' = rN \left(1 - \frac{N}{5000}\right) - 0.01NP = 0 \\
P' = 0.001NP - 0.001P = 0
\end{cases} \Rightarrow \begin{cases} 
N = 0 \text{ or } P = r \left(100 - \frac{N}{50}\right) \\
P = 0 \text{ or } N = 1
\end{cases}
\]

and thus the equilibrium points \((N^*, P^*)\) are \((0, 0)\), \((1, 99.98r)\) and \((5000, 0)\).
Further Exercises 3.4 - 4

(a) The $R$-nullclines are determined to be

$$R' = J - 0.25R^2 = 0 \Rightarrow J = 0.25R^2$$

and the $J$-nullclines are determined to be

$$J' = R + J = 0 \Rightarrow J = -R.$$ 

Figure 3 shows the nullclines of this system.

(b) The equilibrium points $(R^*, J^*)$ are $(0, 0)$ and $(-4, 4)$.

Further Exercises 3.4 - 5

(a) The $R$-nullclines are determined to be

$$R' = 24R - 2R^2 - 3RS = 0 \Rightarrow R = 0 \text{ or } S = 8 - 2R/3$$

and the $S$-nullclines are determined to be

$$S' = 15S - S^2 - 3RS = 0 \Rightarrow S = 0 \text{ or } S = 15 - 3R.$$ 

Figure 4 shows the nullclines of this system.
(b) The equilibrium points \((R^*, S^*)\) are \((0, 0)\), \((0, 15)\), \((12, 0)\) and \((3, 6)\).

**Exercise 3.5.1**

The basin of attraction for \(X = 0\) is \(\{X \mid 0 \leq X < a\}\).

**Exercise 3.5.2**

The basin of attraction for \(X = a\) is \(\{a\}\).

**Exercise 3.5.3**

\(X' = 0\) when the red and blue curves cross.

**Exercise 3.5.4**

The middle equilibrium point is a saddle point.