4. Consider the following Romeo and Juliet model:

\[ R' = J - 0.25R^2 \]
\[ J' = R + J \]

c) Sketch the direction of the change vectors along each nullcline. Then, fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

(-4, 4): unstable

(0, 0): stable
5. Let $R$ be the size of a population of rabbits, and $S$ the population of sheep in the same area. The Lotka–Volterra competition model for these species might look like the following:

$$
R' = 24R - 2R^2 - 3RS \\
S' = 15S - S^2 - 3RS
$$

(c) Sketch the direction of the change vectors along each nullcline. Then, fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

e) How many stable equilibrium points are there? Draw a rough estimate of the basin of attraction of each one. Based on this, what one-word description could you give to this system?

(0, 0): unstable

(12, 0): stable

(0, 15): stable

(3, 6): saddle node
6. Let $D$ be the size of a population of deer, and $M$ the population of moose in the same area. The Lotka-Volterra competition model for these species might look like the following:

$$D' = 0.3D - 0.02D^2 - 0.05DM$$
$$M' = 0.2M - 0.04M^2 - 0.02DM$$

a) Plot the nullclines of this system.

b) Use the nullclines and/or algebra to find the equilibrium points of the system.

c) Sketch the direction of the change vectors along each nullcline. Then fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

e) What will happen to these two populations in the long run? Can they coexist?

$(0, 0)$: unstable
$(0, 5)$: saddle node
$(15, 0)$: stable

The deer and moose population cannot coexist.
7. Repeat the same analysis as in the previous problem, but with the following differential equations:

\[ D' = 0.3D - 0.05D^2 - 0.03DM \]
\[ M' = 0.2M - 0.04M^2 - 0.02DM \]

a) Plot the nullclines of this system.

b) Use the nullclines and/or algebra to find the equilibrium points of the system.

c) Sketch the direction of the change vectors along each nullcline. Then, fill in the change vectors in the rest of the vector field.

d) Use your sketch of the vector field to determine the type of each equilibrium point.

e) What will happen to these two populations in the long run? Can they coexist?

(0, 0): unstable

(0, 5): saddle node

(6, 0): saddle node

(4.29, 2.86): stable

In the long run, deer and moose will coexist, with 4.29 deer and 2.86 moose.
FE 3.5.2

2. How could you use simulation to (approximately) map the basin of attraction of a stable equilibrium?

Simulate using multiple initial conditions. Note: red lines are equilibrium points, blue lines are simulations.

Exercise 3.6.1 Draw phase portraits to confirm what was said about the stabilities of $a$ and $k$, both when $a < k$ and when $a > k$.

$a$ is unstable when $a < k$; $a$ is stable when $a > k$. 
For $a = 600$ and $a = 900$:

- $X = 0$ is stable
- $X = a$ is unstable
- $X = k$ is stable

For $a = 1200$:

- $X = 0$ is stable
- $X = k$ is unstable
- $X = a$ is stable

In one dimension, we cannot have 2 stable or unstable equilibria next to each other, therefore a pair of equilibria produced by a saddle-node bifurcation must consist of one that is stable and one that is unstable.
This is our model:

The blue line stands for growth of budworm, and the red line stands for predation of budworm.

$X = a$ is stable because

- The blue line (growth) is above the red line (predation) to the left of $X = a$, meaning $X' > 0$ before $X = a$
- The red line is above the blue line to the right of $X = a$, meaning $X' < 0$ after $X = a$

$X = b$ is unstable because

- The red line is above the blue line to the left of $X = b$, meaning $X' < 0$ before $X = a$
- The blue line is above the red line to the right of $X = b$, meaning $X' > 0$ after $X = b$

$X = c$ is stable because

- The blue line (growth) is above the red line (predation) to the left of $X = c$, meaning $X' > 0$ before $X = c$
- The red line is above the blue line to the right of $X = c$, meaning $X' < 0$ after $X = c$
Exercise 3.6.6  For each of the \((k, r)\) pairs below, describe how many equilibria the system has, whether they’re high or low, and what their stability is.

a) \(k = 10, r = 0.1\)  
b) \(k = 25, r = 0.6\)  
c) \(k = 20, r = 0.4\)

---

a) \(k = 10, r = 0.1\) falls in the blue region, so there’s only one equilibrium and it’s stable.

b) \(k = 25\) and \(r = 0.6\) falls in the yellow region, so there’s only one equilibrium here. It’s at high \(X\) and stable.

c) \(k = 20\) and \(r = 0.4\) falls in the green (bistable) region, so there are 3 equilibria here. Two are stable and one in the middle is unstable.

---

Top graph corresponds qualitatively to bottom graph as follows (see matching colored regions)
2. The figure below shows a possible relationship between nutrient levels and water turbidity in a lake.

![Figure showing nutrient levels vs. water turbidity](image)

a) If the nutrient level is 0.2, approximately what will the water turbidity level be?

b) If the nutrient level then increases to 0.8, approximately what will the water turbidity level be?

c) Suppose the nutrient level increases further, to 1.0. What will the water turbidity be?

d) You are in charge of water quality for this lake. Your predecessor on the job decided that lowering nutrient levels to 0.8 would be sufficient to restore clear water. What happened to the water turbidity when this was done? Why?

e) How low do nutrient levels need to be for the water to become clear again?

f) The main source of nutrients in the lake is fertilizer washed off from local lawns and gardens. Although people want clear water, significantly reducing fertilizer use is not initially a popular proposal. Explain your nutrient reduction goal in a way community members can understand.

---

a) At nutrient level 0.2, there is one stable equilibrium point at around 0.25. The water turbidity will be 0.25.

b) Starting from a water turbidity level of 0.25, slowly increasing the nutrient level to 0.8 will not leave the basin of attraction of the lower equilibrium point. Therefore, we are at a water turbidity of about 1.5.

c) At nutrient level 1.0, there is no longer a low stable equilibrium point, and so all initial conditions at nutrient level 1.0 will go to the higher equilibrium point. Therefore, at nutrient level 1.0, there will be a water turbidity of about 7.

d) Say clean water is considered turbidity level < 2. If we start from a water turbidity level of 7, then lowering the nutrient level to 0.8 will result in a water turbidity of approximately 5 to 6, which is still high turbidity. This is because decreasing the nutrient level from 1.0 to 0.8 is still within the basin of attraction of the higher equilibrium point.

e) Nutrient levels will have to be reduced past 0.5 for the water levels to become clear again.

f) Change in water turbidity lags behind change in nutrient levels. We need to drastically lower the amount of fertilizer people are using, but only for a short period of time. After obtaining clear water, we can slowly raise the fertilizer levels again.
FE 3.6.3

3. You are studying the effects of psychological stress on movement. Suppose you generated the following bifurcation diagram, where \( r \) is the stress level felt by the subject, and \( X \) is the subject’s muscle tone.

![Bifurcation Diagram](image)

a) List the bifurcations that occur in this diagram. For each one, state what type of bifurcation it is and at what value of \( r \) it occurs.

b) How many stable equilibrium points are there when \( r = 25 \)?

c) Suppose that initially, \( r = 8 \) and \( X = 0.1 \). What happens if \( r \) is increased to 18?

d) What could happen if \( r \) was increased to 22?

- \( r = 5 \): saddle node/blue skies bifurcation
- \( r = 10 \): saddle node/blue skies bifurcation
- \( r = 15 \): saddle node/blue skies bifurcation
- \( r = 20 \): pitchfork bifurcation
- \( r = 30 \): transcritical bifurcation

b) There are 2 stable equilibrium points at \( r = 25 \)

c) \( X \approx 0.5 \) because at \( r = 18 \), there is only 1 stable equilibrium point, which occurs at about \( X = 0.5 \)

d) \( X \) will either increase to about 0.7 or decrease to about 0.4

FE 3.6.5

5. Let \( X \) be the concentration of a certain protein in the bloodstream. The protein is produced at a rate \( f(X) \), and it degrades at a rate \( rX \) (see graphs below). In other words, \( X \) satisfies the differential equation

\[
X' = f(X) - rX
\]

where \( f(X) \) is the function shown in black in the graphs below.

a) Use the “over–under” method to find the equilibrium points of this system, and determine their stability, for the following values of \( r \):
b) Draw a bifurcation diagram for this system as \( r \) varies from 0 to 3. How many bifurcations occur, and what type is each one? You may want to trace or copy the graph of \( f(X) \).
6. Suppose that in the absence of predators, a population grows logistically with $r = 0.75$ and $k = 1$. Also, a fraction $h$ of the population is hunted each year.

a) Write the differential equation for this system.

b) Construct a bifurcation diagram for this system with $h$ as the parameter. What kind(s) of bifurcation(s) do you observe?

a) $X' = 0.75X(1 - X) - hX$