1. Estrogen ($E$), follicle-stimulating hormone ($F$), and gonadotropin-releasing hormone ($G$) have a well-known relationship in the female body. We will make the following assumptions about this system:

- Estrogen is produced by the adrenal glands at a rate of $3 \frac{\text{mg}}{\text{day}}$ per day.
- Estrogen is also produced by the ovaries at a rate proportional to the level of follicle-stimulating hormone ($F$), with a proportionality constant of 0.6.
- Follicle-stimulating hormone ($F$) is produced by the pituitary gland at a rate proportional to the level of gonadotropin-releasing hormone ($G$), with a proportionality constant of 0.4.
- Gonadotropin-releasing hormone ($G$) is produced by the hypothalamus at a rate inversely proportional to (that is, proportional to the reciprocal of) the level of estrogen ($E$), with a proportionality constant of 12.
- Each of the three hormones degrades or is excreted (i.e., leaves the body) at a per-mass rate of 15% per day.

(a) (8 points) Write a system of differential equations to model the levels of these hormones. (As always, a diagram is recommended.)

\[
\begin{align*}
E' &= 3 + 0.6F - 0.15E \\
F' &= 0.4G - 0.15F \\
G' &= \frac{12}{E} - 0.15G
\end{align*}
\]

Question 1 continues on the next page…
(b) (4 points) Identify a feedback loop in this model. What kind of feedback loop is it? Justify your answer. (Hint: A diagram is encouraged.)

The diagram shows a negative feedback loop among all three variables: G signals the pituitary to produce F, which in turn signals the ovaries to produce E, but high levels of E signal the hypothalamus to produce less G, and low levels of E signal the hypothalamus to produce more G.

In short, G increases F, which increases E, which decreases (the production rate of) G.
2. (10 points) Lyme disease is transmitted to humans from ticks. The disease dynamics can be described as follows:

- Susceptible humans \((S)\) are infected at a per-capita rate that is proportional to the density of ticks \((T)\), with a proportionality constant 0.05. They become infected humans \((I)\) once they are infected.
- Susceptible humans die of natural causes or accidents at a constant per-capita rate 0.01.
- Due to the disease, infected people die at the higher per-capita rate of 0.08.
- Infected people who are treated can get better and become susceptible again, but the availability of treatment depends on how many other infected people there are, representing a crowding term. Therefore, infected people become susceptible again at a per-capita rate that is itself proportional to the number of infected people, with a proportionality constant 0.3.
- Infected people can't reproduce, so only susceptible people give birth, at a constant per-capita rate of 0.1. All new babies are born susceptible.
- Tick populations increase according to logistic growth.

Write a system of differential equations for the ticks \((T)\), susceptible \((S)\) and infected humans \((I)\). If you are missing values for any parameters, represent them with a symbol (a letter) and define what each symbol means in words. As always, a diagram is recommended!

\[
\begin{align*}
    S' &= 0.15 - 0.015 - 0.055T + 0.3I^2 \\
    I' &= 0.05ST - 0.3I^2 - 0.08I \\
    T' &= rT \cdot (1 - \frac{I}{k})
\end{align*}
\]

Parameters:  
- \(r\) = natural per-capita growth rate of tick population  
- \(k\) = carrying capacity of tick population

Question 2 continues on the next page...
Question 2 continued...
3. The Beer–Lambert Law states that the intensity of light at a depth $D$ beneath the surface of a body of water is given by 

$$I(D) = I_0e^{-kD},$$

where $I_0$ is the intensity at the surface and $k$ is a positive constant depending on how clear or murky the water is. For the ocean off the coast of Long Beach on a sunny day, $I_0 = 30$ and $k = 0.2$, so that here we have 

$$I(D) = 30e^{-0.2D}.$$ 

While studying kelp there, you measure the depth to be $D = 10 \text{ m}$, and use this to calculate the intensity of the light reaching the kelp bed.

(a) (6 points) Write the linear approximation to $I(D)$ at $D = 10$. (You may write the “short form” or the “long form”, your choice.)

$$\frac{dI}{dD} = 30e^{-0.2D} \cdot (-0.2) = -6e^{-0.2 \cdot 10}$$

$$\left. \frac{dI}{dD} \right|_{D=10} = -6e^{-2} = -0.812$$

**Short form:** 

$$\Delta I \approx (-0.812) \Delta D \quad \text{when} \quad \Delta D \approx 0$$

**Long form:** 

$$I(10) = 30e^{-0.2 \cdot 10} = 30e^{-2} = 4.060$$

$$I(D) \approx 4.060 + (-0.812) \cdot (D - 10) \quad \text{when} \quad D \approx 10$$

(b) (4 points) Suppose that your depth measurement was only accurate to within 4%, so that this measurement may have been off by up to 0.4 m. Use your linear approximation to estimate how far from the correct value your calculation of the light intensity might be.

So, if $\Delta D = 0.4 \text{ m}$, what is $\Delta I$?

$$\Delta I \approx (-0.812) \cdot (0.4) = -0.3248$$

Our calculation of $I$ might be off by about 0.32. Furthermore, if the correct depth is higher than we measured, then our calculation of light intensity would be low by about 0.32, and vice-versa.
4. Romeo and Juliet have a complicated relationship. Their feelings toward each other can be represented by a system of differential equations that tracks the changes in their love (or hate) for each other:

\[
\begin{align*}
R' &= 0.8J \\
J' &= -1.5R \\
\end{align*}
\]

(a) (6 points) Starting at an initial state \( t = 0 \) of love/hate where \( R(0) = 1, J(0) = -1 \), use Euler’s method with a step size \( \Delta t = 0.2 \) to approximate their feelings at \( t = 0.4 \). 

<table>
<thead>
<tr>
<th>( t )</th>
<th>Current state ([ R, J ])</th>
<th>Change vector ([ R', J' ])</th>
<th>Next state</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>([-1, 1])</td>
<td>([0.8, -1.5])</td>
<td>([-1, 1] + 0.2 \times \begin{bmatrix}0.8 \ -1.5\end{bmatrix} = [0.84, -1.3] |</td>
</tr>
<tr>
<td>0.2</td>
<td>([0.84, -1.3])</td>
<td>([-1.04, -1.26])</td>
<td>([0.84, -1.3] + 0.2 \times \begin{bmatrix}-1.04 \ -1.26\end{bmatrix} = [0.632, -1.552] |</td>
</tr>
<tr>
<td>0.4</td>
<td>([0.632, -1.552])</td>
<td>[0.632, -1.552]</td>
<td></td>
</tr>
</tbody>
</table>

At \( t = 0.4 \), the state is approximately \([ R, J ] = \begin{bmatrix}0.632 \\ -1.552\end{bmatrix}\).

(b) (2 points) How would you make this approximation more accurate? 

Make \( \Delta t \) smaller (Closer to 0)
5. (The two parts of this problem are independent of each other.)

(a) (5 points) The graph below shows the instantaneous rate of change of a patient’s blood glucose level $G$ over time. What about the graph tells you the total change in their blood glucose level over a given time interval?

The area between the curve and the $t$-axis over that time interval.

Use the graph to estimate the total change in the patient’s glucose level from $t = 2$ to $t = 4$, using $\Delta t = 0.5$.

Let $f(t) = \frac{dG}{dt}$.

![Graph showing the rate of change $f(t)$ over time $t$.]

\[
\text{Total change} = G(4) - G(2) = \int_{t=2}^{t=4} \left( \frac{dG}{dt} \right) dt = \text{area under curve}
\]

\[
= f(2) \cdot 0.5 + f(2.5) \cdot 0.5 + f(3) \cdot 0.5 + f(3.5) \cdot 0.5
\]

\[
= 1 \cdot 0.5 + 2 \cdot 0.5 + 3 \cdot 0.5 + 3 \cdot 0.5
\]

\[
= 4.5
\]

(Note: Exact area is 5.)

Question 5 continues on the next page...
(b) (5 points) The graph below shows a patient’s insulin level $I$ over time. What about the graph tells you the instantaneous rate of change of their insulin level at a specific time?

The slope of the tangent line to the graph at that time.

Use the graph to find the instantaneous rate of change of their insulin level at $t = 3$.

\[
\text{Slope} = \frac{3-1}{4-2} = \frac{2}{2} = 1
\]
6. In a genetic study, a fraction of mice are given a new gene and allowed to breed with the rest of the population. Let $P$ be the fraction of mice that have the new gene after $t$ weeks. The spread of the gene is modeled by the differential equation

$$ P' = (P - 4P^2)(P^2 - P). $$

(a) (3 points) Find the equilibrium points of this model.

$$ P' = 0; \quad (P - 4P^2)(P^2 - P) = 0 $$

$$ P^2(1 - 4P)(P - 1) = 0 $$

- $P = 0$
- $1 - 4P = 0 \Rightarrow P = \frac{1}{4}$
- $P - 1 = 0 \Rightarrow P = 1$

(b) (5 points) Use the linear stability method (the derivative test) to determine the stability of each equilibrium point, where possible.

Let $f(P) = (P - 4P^2)(P^2 - P)$

Then $\frac{df}{dP} = (1 - 8P)(P^2 - P) + (P - 4P^2)(2P - 1)$

$$ = P(1 - 8P)(P - 1) + P(1 - 4P)(2P - 1) $$

$$ = P(-1 + 9P - 8P^2) + P(-1 + 6P - 8P^2) $$

$$ = P(-2 + 15P - 16P^2) $$

$$ \left. \frac{df}{dP} \right|_{P=0} = 0 \quad \text{so this test is inconclusive for } P^* = 0 $$

$$ \left. \frac{df}{dP} \right|_{P=1} = 1(-2 + 15 - 16) = -3 \not< 0 \quad \text{so } P^* = 1 \text{ is stable} $$

$$ \left. \frac{df}{dP} \right|_{P=\frac{1}{4}} = \frac{1}{4}(-2 + \frac{15}{4} - 1) = \frac{3}{4} > 0 \quad \text{so } P^* = \frac{1}{4} \text{ is unstable} $$

Question 6 continues on the next page...
(c) (3 points) Sketch a phase portrait of this model below, including arrows indicating the direction of the vector field.

(Note that since the eq. point at $P^*=0.5$ is unstable, we know without even having to plug in any test points that the arrows go left (toward $0$). Therefore $0$ is a stable eq. point.)

(d) (3 points) If initially, half of the mice have the new gene, then what portion of the population will end up with the gene in the long run? If only 20% of the mice get the new gene at first, then how many will end up with it in the long run?

If we start at $P(0) = 0.5$, then in the long run, $P(t)$ will go toward 1, meaning eventually all mice will have the gene.

If we start at $P(0) = 0.20$ (to left of $P^*=0.5=0.25$), then $P(t)$ will go toward 0, meaning that eventually none of the mice will have the new gene.
7. Suppose that the number of cells, \( N \), in a bacteria culture is growing at a constant per-capita rate of 6\% per day.

(a) (2 points) Write the differential equation that describes this.

\[
N' = 0.06N
\]

(b) (3 points) Assume that initially, at time \( t = 0 \), there are 400 cells in the culture. Write an expression for \( N(t) \) as a function of \( t \). (That is, give the solution of the differential equation from part (a).)

\[
N(t) = 400e^{0.06t}
\]

(c) (2 points) How many cells will there be after 30 days?

\[
N(30) = 400e^{0.06 \cdot 30} = 2419.9
\]

(d) (3 points) Will this model remain accurate over a long period of time? Why or why not? Explain.

Definitely not! This model predicts unbounded exponential growth, forever! We know that no system can grow exponentially forever. Eventually, this model will begin to differ from real-world observations.
8. Turbidity is the technical term for cloudiness of water, often caused by an abundance of phytoplankton. Scientists have created a mathematical model of the turbidity of a lake, which considers the level of nitrate, a nutrient on which the phytoplankton depend. A bifurcation diagram for this model is shown below.

(a) (3 points) Give a definition of the term bifurcation.

A **bifurcation** is a change in the overall behavior of a model (e.g., change in the number and/or stability of equilibrium points) that occurs as the result of changing the value of a parameter.

(b) (4 points) List the bifurcations that occur in this diagram. For each one, state what type of bifurcation it is and where it occurs.

There are three:

- Pitchfork bifurcation at \( n=1, T=3 \).
- Transcritical bifurcation at \( n=3.5, T=1.5 \).
- Saddle-node bifurcation at \( n=5, T=2 \).

Question 8 continues on the next page...
Question 8 continued... 

(c) (2 points) Under normal circumstances, the nitrate level $n$ is between 2 and 4, and the turbidity is just below 2. Recently, run-off from lawn fertilizers has caused $n$ to increase above 5. What do you expect to happen to the turbidity level?

Since the stable eq. point around $T = 2$ has disappeared (due to the saddle-node bifurcation), the turbidity will increase to the high stable equilibrium, around $T = 5$.

(d) (2 points) What will happen if, after the previous situation, the nitrate level returns to normal (between 2 and 4)?

Without any other external influence to the system, the turbidity will remain at the highest stable equilibrium, which is around $T = 4$ to $T = 5$.

(e) (3 points) As the local land manager, you can set policies that will regulate the amount of nitrate seeping into the lake, and you can use water treatments to temporarily decrease the turbidity of the water, but only by a small amount (up to about 0.5). If the current turbidity is very high (between 4 and 5), what would you do to bring it back to normal levels (a little below 2). Be specific: what actions would you take, and in what order?

(Step 1) Decrease the amount of nitrate to around $n = 1$, and let the system go to equilibrium, at around $T = 3$.

(Step 2) Once we're at $n = 1$ (or slightly less) and $T \approx 3$, use water treatments to decrease $T$ a little, and then immediately increase $n$ to levels greater than 1. This should cause the system to go to the lower stable eq. point at around $T = 2.5$. Then allow $n$ to return to normal.
9. (10 points) A differential equation of the form

\[ X' = [\text{input}] - rX, \]

has the input function shown in the graph to the right. The parameter \( r \) can take either positive or negative values. On the empty axes below, draw a bifurcation diagram for this system, as \( r \) varies.

Note: Be sure to indicate, using your choice of color, pen/pencil, line style, etc, which points in your diagram correspond to stable and unstable equilibria.
10. The following system of differential equations models the populations of mule deer \( (D) \) and mountain lions \( (L) \) in the Angeles National Forest.

\[
\begin{align*}
D' &= 7D - D^2 - DL \\
L' &= D^2L + L - L^2
\end{align*}
\]

(a) (8 points) Sketch the nullclines of this model on the axes below, then use this to obtain a rough sketch of the vector field of the model.

Optional: First change to \( X + Y \):

\[
\begin{align*}
X' &= 7X - X^2 - XY \\
Y' &= X^2Y + Y - Y^2
\end{align*}
\]

(See next page)
(b) (4 points) List all of the equilibrium points of the model, and determine (as best you can) the stability of each one.

<table>
<thead>
<tr>
<th>Eq. point</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>Source (unstable node)</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>Saddle point</td>
</tr>
<tr>
<td>(7, 0)</td>
<td>Saddle point</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>Spiral sink/source/center... can't tell for certain, but it looks like a spiral sink (stable spiral)</td>
</tr>
</tbody>
</table>