1. (10 pts) Kelp (K), sea urchins (U), and sea otters (S) form a food chain off the coast of northern California. Write down a differential equation model of the food chain, by using the following assumptions. In what follows, all rates are per year rates.

- Kelp grows at a per biomass (like per capita) rate of 0.08.
- Due to shading, kelp dies at a per biomass rate proportional to the amount of kelp, with proportionality constant of 0.07.
- Sea urchins eat kelp. A single sea urchin consumes kelp at a rate of 0.06.
- The sea urchin birth rate is proportional to the amount of kelp the urchins (as a whole) consume, with a proportionality constant of 0.05.
- Sea urchins die of natural causes at a per capita rate of 0.04.
- The rate at which a single sea otter eats urchins is proportional to the sea urchin population, with a proportionality constant of 0.03.
- The sea otter birth rate is proportional to the amount of sea urchins the otters (as a whole) consume, with a proportionality constant of 0.02.
- Sea otters die at a per capita rate of 0.01.

**Solution**

\[ K' = 0.08K - 0.07K^2 - 0.06U \]

\[ U' = 0.05(0.06U) - 0.04U - 0.03US \]

\[ S' = 0.02(0.03US) - 0.01S \]
2. (8 points) Mitochondria are organelles that provide energy for human and other eukaryotic cells. Mitochondria can divide like bacteria, and can fuse with each other. Use the following assumptions to write a differential equation for $M$, the number of mitochondria in a cell.

- There is an optimal mitochondria population $m$. The rate at which mitochondria reproduce is proportional to the difference between the current population and the optimal population, with proportionality constant $r$.
- When two mitochondria are close to one another, they may fuse together. Assume this occurs with “probability” $f$.
- Mitochondria die at a constant per-capita rate $d$.

\[
M' = r(m - M) - fM^2 - dM
\]

- reproduction rate
- fusing together decreases the population
- per-capita death rate
3. This coming Winter Quarter, the flu will be spreading among LS 30B students, because all of you didn’t get the flu shot like I told you to. We will model the number of susceptible individuals ($S$) and infected individuals ($I$) by using the differential equations

$$
S' = -0.05SI \\
I' = -0.2I + 0.05SI
$$

a) (2 pts) Explain what each of the two terms in the second equation represents.

**Solution**

The first term represents the rate at which infected individuals either recover or die (!) from the flu.

The second term represents the rate at which susceptible individuals are contracting the flu from infected individuals.


b) (8 pts) Suppose we start with 200 susceptible individuals and 2 infected ones. Use Euler’s method with a step size of 0.1 weeks to determine the approximate numbers of susceptible and infected individuals at $t = 0.2$ weeks.

**Solution**

After 0.2 weeks, $S \approx 194.0796$, $I \approx 7.8012$.

(0.2 weeks is about 34 hours, and the number of infected individuals has nearly quadrupled. I strongly encourage you to run a CoCalc simulation to see what happens in a week... and think about maybe getting that flu shot before you come back to school after the break :)
4. By using the differentiation rules we learned in this course, find \( \frac{df}{dx} \) for each of the following functions \( f(x) \). Indicate clearly which rules you are using at each step of your calculation.

Note: algebraic simplifications will carry no points.

\( a) \) (3 pts) \( f(x) = (x^3 + x^2 + 1)(x^6 + x^4 + 3)(\sqrt{x} + 3x) \)

**Solution**

\[
\frac{df}{dx} = (3x^2 + 2x)(x^6 + x^4 + 3)(\sqrt{x} + 3x) + (x^3 + x^2 + 1)(6x^5 + 4x^3)(\sqrt{x} + 3x) + (x^3 + x^2 + 1)(x^6 + x^4 + 3)\left(\frac{1}{2\sqrt{x}} + 3\right)
\]

by using the Product Rule twice, and the Power Rule.

You are also permitted to use the product rule for three factors:

\[
\frac{d}{dx}(f \cdot g \cdot h) = \left(\frac{df}{dx} \cdot g \cdot h\right) + \left(f \cdot \frac{dg}{dx} \cdot h\right) + \left(f \cdot g \cdot \frac{dh}{dx}\right)
\]

\( b) \) (3 pts) \( f(x) = \ln\left(\frac{x^3 + 1}{x^6 + x^4 + 3}\right) \).

**Solution**

\[
\frac{df}{dx} = \left(\frac{x^6 + x^4 + 3}{x^3 + 1}\right) \left(\frac{(3x^2)(x^6 + x^4 + 3) - (x^3 + 1)(6x^5 + 4x^3)}{(x^6 + x^4 + 3)^2}\right)
\]

by using the Chain Rule, the Logarithmic Rule, the Quotient Rule, and the Power Rule.
By using the differentiation rules we learned in this course, find \( \frac{df}{dx} \bigg|_{x=1} \) for the following function \( f(x) \). Indicate clearly which rules you are using at each step of your calculation.

c) (4 pts) \( f(x) = e^{(x^6 + x^4 + 3)^7} \)

**Solution**

\[
\frac{df}{dx} \bigg|_{x=1} = e^{(x^6 + x^4 + 3)^7} \cdot 7(x^6 + x^4 + 3)^6(6x^5 + 4x^3) \bigg|_{x=1} = e^{5^7} \cdot (7)(5^6)(10) = 1093750e^{78125},
\]

by using the Chain Rule, and the Exponent Rule, and the Power Rule.
5. A solar power plant has a power output of $P(t)$ kW of energy per hour, where $t$ represents time. Suppose that from 6 AM to 12 PM (noon), the power output $P(t)$ is given by

$$P(t) = (t - 6)^2.$$ 

(4 pts) By using time interval $\Delta t = 1$ hour, write down and evaluate the Riemann sum approximating the total energy produced by the power plant over the period from $t = 6$ (6 AM) to $t = 12$ (12 PM).

Solution

$$\sum_{k=0}^{5} ((6 + k) - 6)^2 = \sum_{k=0}^{5} k^2 = 0 + 1 + 4 + 9 + 16 + 25 = 55$$

Thus, the total energy produced is approximately 55 kWh.

b) (4 pts) Write down and evaluate the definite integral that represents the total energy produced by the power plant over the period from $t = 6$ (6 AM) to $t = 12$ (12 PM).

Solution

$$\int_{6}^{12} (t - 6)^2 \, dt$$

Let $F(t) = \frac{1}{3} (t - 6)^3$. Since $F'(t) = (t - 6)^2$, by the FTC,

$$\int_{6}^{12} (t - 6)^2 \, dt = F(12) - F(6) = 72 - 72 = 0$$

Thus, the total energy produced is 72 kWh.

c) (2 pts) Explain why the number your found in part b is larger than the number you found in part a.

Solution

The terms in the Riemann sum represent areas of rectangles drawn under the graph of the function over the subintervals of length 1. Since the height of each rectangle is the value of the function at the left endpoint of the corresponding subinterval, and since the function is increasing, each rectangle underestimates the area under the graph of the function for that subinterval.
7. Suppose you are studying a differential equation model with a single state variable:

\[ X' = f(X) \]

Suppose you know that the model has equilibrium points at \( X = -6 \), \( X = -2 \), \( X = 1 \), and \( X = 5 \), and that

\[ \frac{df}{dX} = X - 0.04X^3. \]

(a) (6 points) What can you say about the stability of each of these equilibrium points? (Hint: Note that I have not given you \( f(X) \), only \( \frac{df}{dX} \).)

\[
\left. \frac{df}{dX} \right|_{X=-6} = -6 - 0.04(-6)^3 = 2.64 > 0 \implies \text{unstable @ } X = -6
\]

\[
\left. \frac{df}{dX} \right|_{X=-2} = -2 - 0.04(-2)^3 = -1.68 < 0 \implies \text{stable @ } X = -2
\]

\[
\left. \frac{df}{dX} \right|_{X=1} = 1 - 0.04(1)^3 = 0.96 > 0 \implies \text{unstable @ } X = 1
\]

\[
\left. \frac{df}{dX} \right|_{X=5} = 5 - 0.04(5)^3 = 0 \quad \text{So we cannot determine the stability of the eq. pt. at } X = 5
\]

(b) (4 points) What is the basin of attraction of each stable equilibrium point that you found in part (a)?

Since we don’t know the stability of \( X = 5 \), we’ll skip that one...

So the only stable one is at \( X = -2 \). Since there are unstable eq. points at \( X = -6 \) and \( X = 1 \), the basin of attraction of the eq. pt. at \( X = -2 \) is the interval \((-6, 1)\), or \(-1 < X < 6\).

Picture:

\[ \text{basin of attraction} \]
8. (8 points) The plot below shows several trajectories of a 2-variable system of differential equations.

List all of the equilibrium points that you can find in this plot, and specify what type each one is. (If you can't tell exactly where one is, you can just approximate.)

There are equilibrium points at (approximately) the following points:

(2, 8): Saddle point (unstable)
(6, 4): Spiral source (or unstable spiral)
(12, 8): Saddle point (unstable)
(20, 8): Sink (or stable node)
8. Romeo and Juliet’s love (or hatred) for each other can be modeled by the following system of differential equations, where $J$ represents Juliet’s love for Romeo (or hatred if negative), and $R$ represents Romeo’s love for Juliet (or hatred if negative):

$$J' = JR - 2J - J^2$$
$$R' = J^2 - R - 4$$

$a$) (4 pts) On the axes provided on the next page, draw the nullclines of this model. Make sure to indicate clearly which ones are the $J$-nullclines, which ones the $R$-nullclines.

Solution

$$J' = J(R - 2 - J) = 0 \Rightarrow J = 0 \text{ or } R = J + 2$$
$$R' = J^2 - R - 4 = 0 \Rightarrow R = J^2 - 4$$

$b$) (3 pts) Identify the equilibrium points of this model in the diagram, and also write down their coordinates here.

Solution

$(-2, 0), (0, -4), (3, 5)$

$c$) (8 pts) By picking a test point from within each region demarcated by the nullclines, determine for each equilibrium point whether it is stable or unstable. Explain briefly the main idea of the method you are employing.

Solution

1) $f(4, 7) = (4, 2)$
2) $f(4, 0) = (-24, 12)$
3) $f(-2, -4) = (8, 4)$
4) $f(-6, 0) = (-24, 32)$
5) $f(-2, 4) = (-8, -4)$
6) $f(2, 6) = (4, -6)$
7) $f(1, 0) = (-3, -3)$
8) $f(-1, 0) = (1, 3)$

By considering the general direction of the change vectors in each of the eight regions demarcated by the nullclines, we see $(3, 5)$ and $(-2, 0)$ are unstable, while $(0, -4)$ is stable: in the first two cases, some of the arrows are pointing away from the points, whereas in the third case, all arrows in the regions adjacent to the equilibrium point are pointing at it.
The arrows are not, and do not need to be, drawn to scale. Remember, we only care about the general direction of the arrows in each segment.
9. Melanin is a pigment in our skin that protects us from skin damage due to sun exposure. A startup biotech company is testing melanin-treatment for skin cancer. They make a mathematical model to design a treatment strategy. In the model’s bifurcation diagram given below, $M^*$ represents the final melanin count at the end of the treatment and $d$ represents the parameter controlling the treatment efficacy. Healthy levels of melanin are between $10^7$ and $10^9$. Answer the following questions and help the company design a clinical trial for test patients with different requirements.

![Bifurcation Diagram]

a) (6 pts) List the bifurcations that occur in this diagram. For each one, state what type of bifurcation it is and at what value of $d$ it occurs.

**Solution**

Saddle-node bifurcations at $d = 20$ ($M^* = 10^7$), $d = 30$ ($M^* = 10^{10}$), and $d = 40$ ($M^* = 10^6$).

(It is enough to note the values of $d$; I have included the values of $M^*$ only for the sake of completeness.)
b) (2 pts) For Patient-1 with an abnormally high initial melanin count of $10^{10}$ and treatment efficacy $d = 10$, can you suggest a strategy to restore normal levels of melanin?

**Solution** Pick $d > 30$.

c) (2 pts) For Patient-2 with an initial low melanin count of $10^6$, the physician chose the treatment efficacy control, $d = 30$, but noticed that this did not restore healthy melanin levels. Why isn’t this treatment working?

**Solution** The state is in the basin of attraction of the low stable equilibrium.

d) (2 pts) For Patient-2, can you help the physician come up with a suitable strategy to restore healthy levels of melanin?

**Solution** Pick $d > 40$. 
5. While studying patients with schizoaffective disorder, you have found that the average dopamine level ($D$) in the patients’ brains can have multiple stable equilibrium points. You have identified a protein that affects the equilibrium levels of dopamine in these patients, and have created a mathematical model that leads to the bifurcation diagram below.

![Bifurcation Diagram]

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(a) (3 points) Give a definition of the term bifurcation.

A bifurcation is a change in the overall (qualitative) behavior of the system due to a change in the value of a parameter. Overall qualitative behavior often means the number and/or type (stability) of equilibrium points.

(b) (5 points) List all of the bifurcations that occur in the diagram above. For each one, state (as specifically as possible) what type of bifurcation it is, and where it occurs.

There are three bifurcations that occur here:

- A saddle-node bifurcation at about $(39, 50)$
- A saddle-node bifurcation at about $(37, 150)$
- A pitchfork bifurcation at about $(50, 90)$

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Question 5 continues on the next page...
(c) (3 points) High average levels of dopamine can lead to hyperactivity and manic episodes, whereas low levels can lead to depression and emotional disconnection. In a healthy person, the average dopamine concentration stabilizes around 90 $\text{pmol/L}$. Fortunately, you have found a treatment that allows you to change the level of the protein ($p$). If presented with a patient for whom $p = 42 \text{ pmol/L}$ and $D$ is too low, how would you need to change $p$ in order to restore $D$ to a healthy level? Likewise, what if $D$ was too high?

Utilize the treatment to decrease $p$ below 30 $\text{pmol/L}$

Then the lower stable equilibrium point will disappear (due to the saddle-node bifurcation), so $D$ must go to the stable equilibrium at $D = 90 \text{ pmol/L}$

If $D$ was too high, do the same, but you'd only need to decrease $p$ to below 37 $\text{pmol/L}$.

(d) (2 points) If, after you administer the treatment in part (c), $p$ returns to 42 $\text{pmol/L}$, will the patient’s dopamine level drop (or increase) again? Why or why not?

No. In the absence of other effects, it would remain at $D = 90 \text{ pmol/L}$, because that is a stable equilibrium point.

(e) (2 points) Explain what sort of change in the level of the protein $p$ might cause the patient’s dopamine level to become abnormally high or low.

If $p$ increases to above 50 $\text{pmol/L}$, the stable eq. pt. at $D = 90 \text{ pmol/L}$ becomes unstable. After that, depending on whether $D$ is slightly above or slightly below 90, $D$ will either increase (to $\sim 180$) or decrease (to $\sim 30$) and stay there.