3.2 Further Exercises

2. The spread of a genetic mutation in a population of mice can be modeled by the differential equation

$$P' = 2P \cdot (1 - P) \cdot (1 - 3P)$$

where $P$ is the fraction of the mice that have the new gene. (This means that $0 \leq P \leq 1$.)

a) Find the equilibrium points of this model and determine the stability of each one.

b) If 10% of the mice have the new gene (so $P = 0.1$) initially, what fraction of the population will have the new gene in the long run?

c) What if the initial fraction is 90% of the mice?

a) To find the equilibria, we must have $P' = 0$. That means that

$$2P \cdot (1 - P) \cdot (3 - P) = 0$$

So we must have

$$\begin{align*}
2P &= 0 & \text{or} & & 1 - P &= 0 & \text{or} & & 1 - 3P &= 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P &= 0 & \text{or} & & P &= 1 & \text{or} & & P &= \frac{1}{3}
\end{align*}$$

So the equilibrium points are 0, $\frac{1}{3}$, and 1. To determine the stability we can use the graphical method. Looking at the plot of $P' = 2P \cdot (1 - P) \cdot (1 - 3P)$ below, we can see that for any value that is not 0 or 1, $P'$ is positive, so it increases towards 1, making 1 a positive equilibrium and 0 an unstable one.

b) If $P$ starts at 0.1, it will eventually increase to $\frac{1}{3}$

c) If $P$ starts at 0.9, it will eventually decrease to $\frac{1}{3}$
To find the equilibrium we must make $L' = 0$, so we get

$$r(k - L) = 0$$

since $r$ is a positive constant, the only way for this condition to be satisfied is if $k - L = 0$, so the only equilibrium is $L = k$. To find the stability, we can use linear stability analysis, and find the derivative of $f(L) = r(k - L)$, which is

$$\frac{df}{dL}(L) = -rL$$

So $\frac{df}{dL}(k) = -rk$ is negative, since $r$ and $k$ are both positive. Therefore it is a stable equilibrium.

We begin by setting $X' = 0$ and solving for $X$

$$X \cdot \left(k - \alpha \ln \left(\frac{X}{X(0)}\right)\right) = 0$$

One equilibrium point is $X = 0$. The second one is found by solving

$$k - \alpha \ln \left(\frac{X}{X(0)}\right) = 0$$

$$\ln \left(\frac{X}{X(0)}\right) = \frac{k}{\alpha}$$

$$\frac{X}{X(0)} = e^{k/\alpha}$$

$$X = X(0)e^{k/\alpha}$$
So the second equilibrium is \( X = X(0)e^{k/\alpha} \). To find the stability of each equilibrium, we can use linear stability analysis and find the derivative of 

\[
\frac{df}{dX}(X) = k - \alpha \ln \left( \frac{X}{X(0)} \right) + X \cdot \frac{d}{dX} \left( k - \alpha \ln \left( \frac{X}{X(0)} \right) \right)
\]

Using the product rule, we have

\[
\frac{df}{dX}(X) = k - \alpha \ln \left( \frac{X}{X(0)} \right) + X \cdot \left( -\alpha \right) \frac{d}{dX} \left( \ln \left( \frac{X}{X(0)} \right) \right)
\]

To find the derivative of the logarithm above, recall that \( \frac{d \ln(X)}{dX} = \frac{1}{X} \) and the chain rule, which gives us

\[
\frac{d}{dX} \left( \ln \left( \frac{X}{X(0)} \right) \right) = \frac{1}{X/X(0)} \cdot \frac{d}{dX} \left( \frac{X}{X(0)} \right)
\]

\[
= \frac{X(0)}{X} \cdot \frac{1}{X(0)}
\]

\[
= \frac{1}{X}
\]

Going back to our original derivative, we have

\[
\frac{df}{dX}(X) = k - \alpha \ln \left( \frac{X}{X(0)} \right) + X \cdot \left( -\alpha \right) \frac{d}{dX} \left( \ln \left( \frac{X}{X(0)} \right) \right)
\]

\[
= k - \alpha \ln \left( \frac{X}{X(0)} \right) - X \cdot \frac{\alpha}{X}
\]

\[
= k - \alpha \ln \left( \frac{X}{X(0)} \right)
\]

Recall that when \( X = 0 \), then \(-\alpha \ln \left( \frac{X}{X(0)} \right)\) becomes very large (tends to infinity), so \( \frac{df}{dX}(0) \) is positive, so 0 is unstable. On the other hand,

\[
\frac{df}{dX}(X(0)e^{k/\alpha}) = k - \alpha - \alpha \ln(e^{k/\alpha})
\]

\[
= k - \alpha - \frac{k}{\alpha}
\]

\[
= -\alpha
\]

So \( X = X(0) \) is stable. Therefore, the tumor will eventually grow to the stable equilibrium, \( X(0)e^{k/\alpha} \).
7. Is it possible for a one-dimensional system to have two stable equilibria without an unstable one between them? Explain. (Hint: Try drawing the situation.)

This is not possible. One way to see this is through the graphical representation of the linear stability analysis. A stable equilibrium happens when the graph of $X'$ versus $X$ crosses the x-axis with a downward tangent line (green dots in the figure below), while an unstable equilibrium happens when the graph crosses the x-axis with an upward tangent line (red dot in the figure below). It would be impossible to draw the graph of a function to cross the x-axis going down twice, without coming back up.

3.3 In-Text Exercises

**Exercise 3.3.1** Find the equilibrium point of the system of equations $X' = -0.5X, Y' = -Y$.

To find the equilibrium, we set $X' = 0$ and $Y' = 0$, so we must solve the system

$$
-0.5X = 0 \\
-Y = 0
$$

So the equilibrium is given by the point $(0, 0)$.

**Exercise 3.3.2** What is the change vector at the point $(3, -4)$?

The change vector at the point $(3, -4)$ is given by $(X', Y') = (X, Y) = (3, -4)$
Both time series plots below correspond to trajectories on the first picture \((X' = Y, Y' = -X)\). The first one corresponds to the first quadrant (positive \(X\) and \(Y\)), while the second one corresponds to the second quadrant (positive \(X\) and negative \(Y\)).

### 3.4 In-Text Exercises

**Exercise 3.4.1** Find the equilibria for the shark–tuna model

\[
\begin{align*}
S' &= 0.01ST - 0.2S \\
T' &= 0.05T - 0.01ST
\end{align*}
\]

To find the equilibria we must have both \(S'\) and \(T'\) equal to zero. By making \(S' = 0\), substituting the derivative and factoring out \(S\) we get
\[ 0.01ST - 0.2S = 0 \]
\[ S(0.01T - 0.2) = 0 \]

So \( S' = 0 \) if, and only if, \( S = 0 \) or \( 0.01T = 0.2 \Rightarrow T = 20 \). Similarly, if we make \( T' = 0 \) we can substitute the derivative and factor out the \( T \) to get

\[ 0.05T - 0.01ST = 0 \]
\[ T(0.05 - 0.01S) = 0 \]

So \( T' = 0 \) only if \( T = 0 \) or \( 0.05 - 0.01S = 0 \Rightarrow S = 5 \). We can see then, that the only points that make both derivatives zero at the same time are \((0,0)\) and \((5, 20)\).

**Exercise 3.4.2** Find the \( M \)-nullcline for the first deer-moose competition model, \( D' = D(3 - M - D), \ M' = M(2 - M - 0.5D) \).

To find the \( M \)-nullcline we must make \( M' = 0 \). This leads us to the equation

\[ M(2 - M - 0.5D) = 0 \]

This equation is satisfied only if \( M = 0 \) or if \( 2 - M - 0.5D = 0 \). If we consider the phase space to be a 2 dimensional Cartesian plane, where \( D \) is on the x-axis and \( M \) is on the y-axis, then the condition \( M = 0 \) is satisfied over the x-axis. The condition \( 2 - M - 0.5D = 0 \) can be rearranged into \( M = 2 - 0.5D \), which corresponds to the straight line with slope 0.5 and y- intercept 2. Taken together, these two conditions define the nullcline shown below.
Equilibria happen only when the derivatives of all state variables in the system are zero. Since the nullclines represent the points when the derivative of each state variable is zero, the only points that can be equilibria are the intersection of state variables.

**Exercise 3.4.3** Why do equilibria occur where nullclines cross? Can they occur anywhere else?

**Exercise 3.4.4** Find the $M$-nullclines for this model.

The $M$-nullclines are the regions where $M' = 0$. Substituting the value of the derivative we get

\[
2M - DM - M^2 = 0 \\
M(2 - D - M) = 0
\]

Therefore the $M$-nullclines are given by $M = 0$ and $2 - D - M = 0$, which can rewritten as $M = 2 - D$.

**Exercise 3.4.5** Find the model’s equilibria.

The models equilibria are the intersection of the nullclines. Three of the equilibria are easily found by setting either $M$ or $D$ (or both) to zero, which gives us $(0, 0)$, $(0, 2)$, $(3, 0)$. The fourth equilibria is found by solving the system

\[
M = -\frac{1}{2}D + \frac{3}{2} \\
M = 2 - D
\]

We can solve this by substituting $D$, which gives us

\[
-\frac{1}{2}D + \frac{3}{2} = 2 - D \\
\frac{1}{2}D = \frac{1}{2} \\
D = 1
\]

By substituting this value back in one of the original equations we get $M = 2 - 1 = 1$. So the final equilibrium is $(1, 1)$.

**Exercise 3.4.6** Use this procedure to sketch the change vectors on the $M$-nullclines.
If we begin by the $M = 0$ portion of the $M$-nullcline, we can take test points of either side of the $(3, 0)$ equilibrium

- Test point 1: $(D', M')_{(2, 0)} = (2, 0)$ change vector points right
- Test point 2: $(D', M')_{(4, 0)} = (-4, 0)$ change vector points left

We can also do this for points in the $M = 2 - D$ portion of the $M$-nullcline, on either side of the $(1, 1)$ equilibrium

- Test point 3: $(D', M')_{(0.5, 1.5)} = (-0.25, 0)$ change vector points left
- Test point 4: $(D', M')_{(1.5, 0.5)} = (0.75, 0)$ change vector points right

**Exercise 3.4.7** What is the biological significance of the fact that this equilibrium is a saddle point?

This means that the equilibrium is unstable, and the system will go towards one of the equilibria where only one species is able to coexist. Which equilibrium that is will depend on the initial conditions of the system.

**Exercise 3.4.8** Find the nullclines and equilibrium points of the Lotka–Volterra predation model, $N' = 0.05N - 0.01NP$, $P' = 0.005NP - 0.1P$. Then, sketch the vector field.

To find the $N$-nullcline we make $N' = 0$, which gives us

\[
0.05N - 0.01NP = 0
\]

\[
N(0.05 - 0.01P) = 0
\]

So the $N$-nullcline is given by $N = 0$ and $0.05 - 0.01P$, which can be rewritten as $P = 5$. Similarly, the $P$-nullcline is found by making $P' = 0$ which gives us

\[
0.005NP - 0.1P = 0
\]

\[
P(0.005N - 0.1) = 0
\]

So the $P$-nullcline is given by $P = 0$ and $N = 20$. By plotting the nullclines we see that there are only two points of intersection $(0, 0)$ and $(20, 5)$. 

8
If we make $M' = 0$, then we have

$$M(r_M - k_M D - c_M M) = 0$$

The $M$-nullcline is given by $M = 0$ or $r_M - k_M D - c_M M = 0$, which can be rewritten as $M = \frac{r_M}{c_M} - \frac{k_M}{c_M} D$.

### 3.4 Further Exercises

1. Consider the Lotka–Volterra predation model, $N' = rN - aNP$, $P' = cNP - \delta P$, with $N$ the number of prey and $P$ the number of predators.
   a) Without doing any algebra, explain why there are no equilibria at which one species has a nonzero population and the other does not.
   b) Find the equilibria.

a) This model assumes the prey has exponential growth in the absence of the predator and that the predator is a specialist and depends on the prey to survive. Therefore, in the absence of prey, the predators cannot sustain a positive population and go extinct, and in the absence of predators the prey grow exponentially and do not reach an equilibrium.

b) We must have both derivatives be zero at the same time. Let’s begin by making $N' = 0$, that gives us

$$rN - aNP = 0$$

$$N(r - aP) = 0$$
So we must have $N = 0$ or $P = \frac{r}{a}$.

By making $P' = 0$ we get

\[ caNP - \delta P = 0 \]
\[ P(caN - \delta) = 0 \]

So we must have either $P = 0$ or $N = \frac{\delta}{ca}$. Therefore the only possible equilibria are $(0, 0)$ and $(\frac{r}{a}, \frac{\delta}{ca})$.

By making $N' = 0$ we have

\[ 0.1N\left(1 - \frac{N}{5000}\right) - 0.01NP = 0 \]
\[ N\left(0.1 - \frac{N}{50000} - 0.01P\right) = 0 \]

So the $N$-nullclines are given by $N = 0$ and $0.1 - \frac{N}{50000} - 0.01P = 0$, which can be rearranged as $N = 5000 - 500P$. For the $P$-nullclines, if we make $P' = 0$ we get

\[ 0.001NP - 0.001P = 0 \]
\[ P(0.001N - 0.001) = 0 \]

so the $P$-nullclines are $P = 0$ and $0.001N - 0.001 = 0$, which can be rearranged as $N = 1$. If we set $P = 0$ we get two possible equilibria, which are $(0, 0)$, $(5000, 0)$. There is a third equilibrium, obtained by setting $P = 1$ which gives us $N = 4500$, so the equilibrium is $(4500, 1)$.  

2. The growth of a population in the absence of predators is described by the logistic equation with $r = 0.1$ and $K = 5000$. To model the predation, we add a term representing the consumption of prey by the predators. We assume that a single predator consumes prey at a per prey individual rate of 0.01. We also assume that the contribution of the prey to the predator birth rate is small, and has coefficient 0.001, and that the predator per capita death rate is 0.001. If the prey population size is $N$ and the predator population size is $P$, we have the differential equations $N' = rN\left(1 - \frac{N}{5000}\right) - 0.01NP$, $P' = 0.001NP - 0.001P$. Find the equilibria of this system.

By making $N' = 0$ we have

\[ 0.1N\left(1 - \frac{N}{5000}\right) - 0.01NP = 0 \]
\[ N\left(0.1 - \frac{N}{50000} - 0.01P\right) = 0 \]

So the $N$-nullclines are given by $N = 0$ and $0.1 - \frac{N}{50000} - 0.01P = 0$, which can be rearranged as $N = 5000 - 500P$. For the $P$-nullclines, if we make $P' = 0$ we get

\[ 0.001NP - 0.001P = 0 \]
\[ P(0.001N - 0.001) = 0 \]

so the $P$-nullclines are $P = 0$ and $0.001N - 0.001 = 0$, which can be rearranged as $N = 1$. If we set $P = 0$ we get two possible equilibria, which are $(0, 0)$, $(5000, 0)$. There is a third equilibrium, obtained by setting $P = 1$ which gives us $N = 4500$, so the equilibrium is $(4500, 1)$.
4. Consider the following Romeo and Juliet model:

\[ R' = J - 0.25R^2 \]
\[ J' = R + J \]

a) Plot the nullclines of this system. (Recall that both \( R \) and \( J \) can be negative!)

b) Use the nullclines and/or algebra to find the equilibrium points of the system.

a) The \( R \)-nullcline is given by \( J - 0.25R^2 = 0 \), which can be rewritten as \( J = 0.25R^2 \) which is a parabola. The \( J \)-nullcline is given by \( R + J = 0 \). The plot of the nullclines is

b) From the nullclines we can see that there are two equilibria, which are the points \((0,0)\) and \((-4,4)\). This can be confirmed by plugging those numbers into the nullclines equations above.

5. Let \( R \) be the size of a population of rabbits, and \( S \) the population of sheep in the same area. The Lotka–Volterra competition model for these species might look like the following:

\[ R' = 24R - 2R^2 - 3RS \]
\[ S' = 15S - S^2 - 3RS \]

(Refresh your memory about what each of the six terms in the equations above represents.)

a) Plot the nullclines of this system.

b) Use the nullclines and/or algebra to find the equilibrium points of the system.

a) To find the \( R \)-nullclines we make
\[ 24R - 2R^2 - 3RS = 0 \]
\[ R(24 - 2R - 3S) = 0 \]

So the \( R \)-nullclines are given by \( R = 0 \) and \( 24 - 2R - 3S = 0 \), which can be rewritten as \( S = 8 - \frac{2}{3}S \). Similarly to find the \( S \)-nullclines we make

\[ 15S - S^2 - 3RS = 0 \]
\[ S(15 - S - 3R) = 0 \]

So the \( S \)-nullclines are given by \( S = 0 \) and \( 15 - S - 3R \), which can be rewritten as \( S = 15 - 3R \). The nullcline plot then is as below.

b) The equilibria can be found in the nullcline plot, and are \((0, 0), (12, 0), (0, 15), (3, 6)\). this can be confirmed by plugging those values into the nullcline equations above.

3.5 In-Text Exercises

**Exercise 3.5.1** What is the basin of attraction for \( X = 0 \)?

The basin of attraction for \( X = 0 \) is all points smaller than \( a \).

**Exercise 3.5.2** Does \( X = a \) belong to either basin of attraction? \( (\text{Hint: Where does a system starting exactly at } X = a \text{ go?}) \)

\( X = a \) does not belong to either basin of attraction, as it is an equilibrium point. Any trajectory starting at \( a \) will remain there.
Exercise 3.5.3  How is the lactose concentration changing when the red and blue curves cross?
When the red and blue curves cross there is no change to the level of lactose. The inflow and outflow of lactose are equal, therefore the rate of change is zero.

Exercise 3.5.4  In Figure 3.37, what kind of equilibrium point is the middle one?
A saddle equilibrium.