3.4. FE 4 c, d
5 c, d, e
6
7

3.6. In text
FE 1
2
3
5
6 ab

3.4. FE 4

R nullcline: \( J = 0.25R^2 \)

\( J \) nullcline: \( J = -R \)

Remember to pick points on nullclines and on either side of eq. points

A: unstable

B: saddle
3.4. FE 5

\[ R' = 24R - 2R^2 - 3RS = R \cdot (24 - 2R - 3S) \]
\[ S' = 15S - S^2 - 3RS = S \cdot (15 - S - 3R) \]

**R-nullcline:** \( R' = 0 \rightarrow R = 0 \) or \( S = \frac{1}{3} \cdot (24 - 2R) \)

**S-nullcline:** \( S' = 0 \rightarrow S = 0 \) or \( S = 15 - 3R \)

---

**Diagram:**

- **Basin of Attraction for A**
- **Approx Division Line**
- **Basin of Attraction for D**

---

Remember that eq points occur where nullclines intersect (red crosses blue).

**A:** stable

**B:** saddle

**C:** unstable

**D:** stable

There are 4 eq points. \( R \) and \( S \) cannot coexist as the system will go towards **A** or **D** in the long run.
\[ D' = 0.3D - 0.02D^2 - 0.05DM = D(0.3 - 0.02D - 0.05M) \]

\[ M' = 0.2M - 0.04M^2 - 0.02DM = M(0.2 - 0.04M - 0.02D) \]

**D-nullcline:** \[ D' = 0 \rightarrow D = 0 \text{ or } 0.3 - 0.02D - 0.05M = 0 \]

\[ L: D = 15 - \frac{5}{2} \cdot D \cdot M \]

**M-nullcline:** \[ M' = 0 \rightarrow M = 0 \text{ or } 0.2 - 0.04M - 0.02D = 0 \]

\[ L: D = 10 - 2M \]

- \( D \text{-nullcline} \)
- \( M \text{-nullcline} \)

**A:** Stable  \( \quad \) **B:** Unstable  \( \quad \) **C:** Saddle
Part c:

$$D' = 0 \rightarrow D = 0 \quad \text{or} \quad D = 15 - \frac{5}{2} M \quad \{2 \text{ cases}\}$$

$$M' = 0 \rightarrow M = 0 \quad \text{or} \quad D = 10 - 2M \quad \{2 \text{ cases}\}$$

- **Case #1**: \( D = 0 \) and \( M = 0 \)

  \[ \text{Eq. point: (0,0)} \]

  well... that was easy

- **Case #2**: \( D = 0 \) and \( D = 10 - 2M \)

  \[ \Rightarrow \ O = 10 - 2M \rightarrow M = 5 \]

  \[ \text{Eq. point: (5,0)} \]

- **Case #3**: \( D = 15 - \frac{5}{2} M \) and \( M = 0 \)

  \[ \Rightarrow \ D = 15 - \frac{5}{2} \cdot 0 = 15 \]

  \[ \text{Eq. point: (0,15)} \]

- **Case #4**: \( D = 15 - \frac{5}{2} M \) and \( D = 10 - 2M \)

  You can tell from graph that this solution does not exist within our state space.

  But if you want to solve:

  \[
  D = 15 - \frac{5}{2} M \\
  D = 10 - 2M
  \]

  \[ \Rightarrow 15 - \frac{5}{2} M = D = 10 - 2M \rightarrow 15 - \frac{5}{2} M = 10 - 2M \rightarrow M = 10 \]

  \[ \text{Eq pt: (10, -10)} \]

  Not in our state space

Part e:

They cannot coexist as they approach the state \( M = 0, D = 15 \) in the long run.
$D' = 0.3D - 0.05D^2 - 0.03DM = D (0.3 - 0.05D - 0.03M)$

$M' = 0.2M - 0.04M^2 - 0.02DM = M (0.2 - 0.04M - 0.02D)$

$D$-nullcline: $D = 0$ or $0.3 - 0.05D - 0.03M = 0$

$\rightarrow D = 6 - \frac{3}{8}M, \frac{3}{5}M$

$M$-nullcline: $M = 0$ or $0.2 - 0.04M - 0.02D = 0$

$\rightarrow D = 10 - 2M$

---

**Graphical Diagram**

- **D** nullcline
- **M** nullcline

**Points and Classification**:

- **A**: saddle
- **B**: unstable
- **C**: saddle
- **D**: stable
Part b:

we have: \( D' = 0 \rightarrow D = 0 \) or \( D = 6 - \frac{3}{5} M \)

\( M' = 0 \rightarrow M = 0 \) or \( D = 10 - 2M \)

\( \Rightarrow 4 \) possibilities

Case #1:

\( D = 0; \ M = 0 \)

Case #2:

\( D = 0 \) \& \( D = 10 - 2M \) \rightarrow \( D = 0 \) and \( M = 5 \)

Case #3:

\( D = 6 - \frac{3}{5} M \) \& \( M = 0 \) \rightarrow \( D = 6 \) and \( M = 0 \)

Case #4:

\( D = 6 - \frac{3}{5} M \) \& \( D = 10 - 2M \)

\( \rightarrow 6 - \frac{3}{5} M = 10 - 2M \) \rightarrow \( M = \frac{10 - 6}{2 - \frac{3}{5}} = \frac{4 \cdot 5}{10 - 3} = \frac{20}{7} \)

\( \rightarrow D = 10 - 2M = 10 - 2 \cdot \frac{20}{7} = \frac{30}{7} \)

Part e:

Yes, they can as the system goes towards \( D = \frac{30}{7}; \ M = \frac{20}{7} \)

in the long run
when $a < k$

<table>
<thead>
<tr>
<th>Sign of</th>
<th>Section #1</th>
<th>Section #2</th>
<th>Section #3</th>
<th>Section #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r \cdot X$</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>$1 - \frac{X}{k}$</td>
<td>(+)</td>
<td>(+)</td>
<td>(+)</td>
<td>(-)</td>
</tr>
<tr>
<td>$\frac{X}{a} - 1$</td>
<td>(-)</td>
<td>(-)</td>
<td>(+)</td>
<td>(+)</td>
</tr>
<tr>
<td>$X'$</td>
<td>(+)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

when $a > k$

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<td>(+)</td>
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<td>$X'$</td>
<td>(+)</td>
<td>(-)</td>
<td>(+)</td>
<td>(-)</td>
</tr>
</tbody>
</table>

How to read solutions:

For each section, examine each term if it is (+) or (-)

→ deduce the (+)/(-) for the overall $X'$ for that section
we all know that for the equation: \( x' = r x \left(1 - \frac{x}{k}\right)\left(\frac{x}{a} - 1\right) \)

we have eq points at: \( x = 0, \ x = k \) & \( x = a \)

we have \( k = 1000 \)

\( a \) is changing

\( a = 600: \)

\( x = 0: \) Stable

\( x = 600: \) unstable \((= a)\)

\( x = 1000: \) stable \((= k)\)

\( a = 900: \)

\( x = 0: \) stable

\( x = 900: \) unstable \((= a)\) \((= a)\)

\( x = 1000: \) stable \((= a)\) \((= k)\)

\( a = 1200: \)

\( x = 0: \) stable

\( x = 1000: \) unstable \((= k)\)

\( x = 1200: \) stable \((= a)\)
Yes. As you saw from previous sections, it is not possible for two eq points of the same stability next to each other, as it is not possible for a vector to change direction as it goes from the eq point of the smaller value to the eq point of the larger value without passing through an eq point of opposite stability.

Graphically,

Need an unstable eq pt

Need a stable eq point
3.6.4

Low $K$

High $K$, same $r$

High $K$, low $r$

High $K$, high $r$
Example code:

```python
@ interact

def spruce(r=(0, 1, 0.1)):
    k = 10
    p1 = plot(r*x*(1-(x/k)), (x, 0, 11), ymin=0, ymax=3)
    p2 = plot((x**2)/(1+(x**2)), (x, 0, 11), color="red")
    p = p1 + p2
    show(p)
```

At \( r=0 \), 1 eq point at \( x=0 \)

For low \( r \) (between 0 and \( \approx 0.3 \)), 2 eq points (one of which is \( x=0 \))

For med \( r \) (between \( \approx 0.3 \) and \( \approx 0.6 \)), 4 eq points (one of which is \( x=0 \))

For high \( r \) (above \( \approx 0.6 \)), 2 eq points (one of which is \( x=0 \))
Refuge: stable at low density

Outbreak: stable at high density

Note: $x=0$ is always an existing unstable eq point in this system

Part a:

$k = 10$, $r = 0.1 \rightarrow$ In refuge - only region

$\rightarrow 2$ eq points $\rightarrow \left\{ \begin{array}{l} x=0: \text{an unstable pt} \\ \text{a refuge point, stable at low density} \end{array} \right.$

Part b:

$k = 25$, $r = 0.6 \rightarrow$ In outbreak - only region

$\rightarrow 2$ eq points $\rightarrow \left\{ \begin{array}{l} x=0: \text{an unstable point} \\ \text{an outbreak point, stable at high density} \end{array} \right.$

Part c:

$k = 20$, $r = 0.4 \rightarrow$ In bistable region

$\rightarrow 4$ eq points $\rightarrow \left\{ \begin{array}{l} x=0: \text{an unstable point} \\ \text{a refuge point, stable at low density} \\ \text{an unstable eq pt at intermediate density} \\ \text{an outbreak point, stable at high density} \end{array} \right.$
Recommend picking: 1 value for $0 < a < 1$

2 values for $a > 1$

Pick any value for $v$

For example:

$v = 2, \ a = 0.5$

Eq pt at: $x = 0$ (stable)

$v = 2, \ a = 2$

Eq pts at: $x = -0.968$ (stable)

$x = 0$ (unstable)

$x = 0.968$ (stable)

$v = 2, \ a = 4$

Eq pts at: $x = -0.999$ (stable)

$x = 0$ (unstable)

$x = 0.999$ (stable)
It does not make biological sense as:

- threshold implies unstable
  
  Below threshold → goes to a certain value
  
  Above threshold → goes towards another value
  
  (e.g. think action potential) - system does not stay at threshold

- Carrying capacity implies long-term stability

It will behave reasonably when we have: $0 < A < K$
Part a:
Turbidity ≈ 0.4

Part b:
Original state ≈ 0.4
New turbidity ≈ 1.6 → water turbidity increases to ~ 1.6

Part c:
Original state ≈ 1.6
New turbidity ≈ 6.5 → water turbidity increases to ~ 6.5

Part d:
Original state ≈ 6.5
New turbidity ≈ 5.5 → water turbidity decreases to ~ 5.5

It stabilizes around the higher value since the original state was so high up that it is in the basin of attraction of the upper eq pt.
Part e:
It has to be less than 0.5 to get off the basin of attraction of the upper eq pts.

Part f:
Between a certain nutrient levels (0.5 to 1), the water turbidity tends to stabilize at two different levels, depending on the original turbidity level.

It makes sense that if it starts out too high, it will stabilize to the higher level. And if it starts out low, it will stabilize at the lower level.

So the plan is to initially reduce the use of fertilizer to clear off the water. Thus, when we start increasing the amount of fertilizer, the system starts at low turbidity level, and it will stabilize at a lower value.
Part a:

$r = 5$: saddle node

$r = \frac{20}{\pi}$

$r = 10$: saddle node

$r = 15$: saddle node

$r = 20$: pitchfork

$r = 30$: transcritical

Part b:

There are 2

Part c:

$x$ will increase to $\sim 0.5$

Part d:

$x$ will decrease to $\sim 0.4$

("Original state" is in the basin of attraction of the lower stable eq. pt.)

("Original state" is the end result obtained in part c, and becomes the "original" for part d)
Part a:

\[ x \approx 4.5 \text{ (stable)} \]

\[ r = 0.4 \]

\[ x \approx 1.8 \text{ (stable)} \]
\[ x \approx 2.5 \text{ (unstable)} \]
\[ x \approx 4.2 \text{ (stable)} \]

\[ r = 0.8 \]

\[ x \approx 1.5 \text{ (stable)} \]
\[ x \approx 3 \text{ (unstable)} \]
\[ x \approx 3.5 \text{ (stable)} \]

\[ r = 1.1 \]

\[ x \approx 1.2 \text{ (stable)} \]

\[ r = 2 \]

---

Part b:

we have 2 bifurcation points
(r = 0.7 and r = 1.2), and both are saddle node bifurcations
Part a:

\[ X' = rX \left(1 - \frac{X}{K}\right) - h \cdot X = 0.75X \left(1 - \frac{X}{1}\right) - h \cdot X \]

\[ \rightarrow X' = 0.75X (1 - X) - h \cdot X \]

Part b:

If we graph the input and output components of \( X' \)
The bifurcation diagram is as follows:

\[ \text{---: stable} \]
\[ \text{--: unstable} \]

It looks like a transcritical bifurcation (though not 100% the same as in the definition).