Exercise 1 (Linear transform of exponential RV). Let \( X \sim \text{Exp}(\lambda) \) and fix constants \( a, b \in \mathbb{R} \) with \( a \neq 0 \). Show that
\[
f_{aX+b}(x) = \frac{\lambda}{|a|} e^{-\lambda(x-b)/a} 1((x-b)/a > 0).
\] (1)
Is \( aX + b \) always an exponential RV?

Exercise 2 (Cauchy from uniform). (Hint: Prop. 1.7 in LN1) Let \( X \sim \text{Uniform}((-\pi/2, \pi/2)) \). Define \( Y = \tan(X) \).

(i) Show that \( d\tan(y)/dy = \sec^2(y) \).

(ii) Show that \( 1 + \tan^2(y) = \sec^2(y) \). (Hint: draw a right triangle with angle \( y \))

(iii) Recall that arctan is the inverse function of tan. Show that \( \text{arctan}(t) \) is strictly increasing and differentiable. Furthermore, show that
\[
\frac{d}{dt} \text{arctan}(t) = \frac{1}{1+t^2}.
\] (2)

(iv) Show that \( Y \) is a standard Cauchy random variable, that is,
\[
f_Y(y) = \frac{1}{\pi(1+y^2)}.
\] (3)

Exercise 3. Let \( X, Y \sim \text{Uniform}([0,1]) \) be independent uniform RVs. Define \( Z = X + Y \). Observe that the pair \( (X, Y) \) is uniformly distributed over the unit square \([0,1]^2\). So
\[
\mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) = \text{Area of the region \{(x, y) \in [0,1]^2 | x + y \leq z\}}.
\] (4)

(i) Draw a picture shows that
\[
\mathbb{P}(Z \leq z) = \begin{cases} 
z^2/2 & \text{if } 0 \leq z \leq 1 \\
1 - (2-z)^2/2 & \text{if } 1 \leq z \leq 2 \\
0 & \text{otherwise.}
\end{cases}
\] (5)

(ii) Conclude that
\[
f_Z(z) = \begin{cases} 
z & \text{if } 0 \leq z \leq 1 \\
2-z & \text{if } 1 \leq z \leq 2 \\
0 & \text{otherwise.}
\end{cases}
\] (6)

Exercise 1.11. Let \( X_1 \sim \text{Exp}(\lambda_1) \) and \( X_2 \sim \text{Exp}(\lambda_2) \) and suppose they are independent. Define \( Y = \min(X_1, X_2) \). Show that \( Y \sim \text{Exp}(\lambda_1 + \lambda_2) \). (Hint: Compute \( \mathbb{P}(Y \geq y) \).)

Exercise 1.15 (Sum of ind. Poisson RVs is Poisson). Let \( X \sim \text{Poisson}(\lambda_1) \) and \( Y \sim \text{Poisson}(\lambda_2) \) be independent Poisson RVs. Show that \( X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2) \).