Midterm 1
Version A

Last Name: ________________________________
First Name: ______________________________
Last six digits of UID: ______________________

By signing below, you affirm that you have neither given nor received unauthorized help on this exam.

Signature: ________________________________

Instructions: Do not open this exam until instructed to do so. You will have 90 minutes to complete the exam. Please print your name and the last six digits of your student ID number above. You may not use books, notes, or any other material to help you. You may use a calculator, but not a programmable or graphing calculator. Please make sure your phone is silenced and stowed with your other belongings at the front of the room. You may use any available space on the exam for scratch work, including the backs of the pages. If you need more scratch paper, please ask one of the proctors.

Please do not write below this line.

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1. (10 points) In a viral infection, some number of viruses \((V)\) infect the cells of a host organism, which produces antibodies \((A)\) to fight the infection. The number of infected cells will be called \(I\). Write a differential equation model to describe this, based on the following assumptions:

- When a cell is infected, the virus begins producing copies of itself within the cell, which are released when the cell dies. This takes some time, so the rate at which new viruses are \emph{produced} is proportional to the number of infected cells 2 hours earlier.

- Production of antibodies is triggered by the immune system in response to the presence of viruses. But as with many physiological responses, this takes some time. So the rate at which antibodies are produced is an increasing sigmoid function of the number of viruses that were present 3.5 hours earlier, with a maximum of 260 per hour.

- Antibodies act to prevent the infection of new cells, so the infection rate, \emph{per capita of viruses}, is highest when there are very few antibodies, and drops off steeply when there are many antibodies present.

- Free viruses don’t last long on their own, so the per-capita death rate of the viruses is 150\% per hour.

- The antibodies decay at a rate of 7\% per hour.

- The infected cells die at a rate of 35\% per hour.

As usual, it is recommended that you start with a diagram. You may find that you need to include some parameters in your model that are not specified here. If you do, give a brief description of what each parameter represents.

Question 1 continues on the next page...
Question 1 continued...  

Last six digits of UID: _____________
2. (10 points) In the skies above Westwood, peregrine falcons prey on recently hatched young in crows’ nests. Using the following assumptions, write a system of differential equations for the population of baby (hatchling) crows ($C$) and the population of peregrine falcons ($F$).

- Because it takes a crow 2 years to reach sexual maturity and they have a lifespan of 9 years, the per-capita birth rate of the crows is proportional to the number of hatchlings 4 years earlier, with a proportionality constant of 2.8.

- Each falcon preys on crow hatchlings at a rate that increases with the abundance of baby crows, up to a maximum (saturation) level of 1.3.

- Because they have many food sources, even in the absence of crows, the falcon population will grow logistically with a natural per-capita growth rate of 6% per year, and a carrying capacity of 220.

- Although the falcon population is not significantly affected by a lack of baby crows, when there is an abundance of baby crows, more falcons may be born, up to an additional per-capita birth rate of as much as 4% per year. (Use a steeply increasing sigmoid function for this, with a maximum of 4%.)

- Baby crows have a per-capita death rate of 18% per year. In addition to that, about 40% of them reach maturity each year. (Note that when they reach maturity, they are no longer baby crows.)

As usual, it is recommended that you start with a diagram. You may find that you need to include some parameters in your model that are not specified here. If you do, give a brief description of what each parameter represents.
Question 2 continued... Last six digits of UID: _____________
4. (a) (4 points) Give a definition of the term \textit{attractor}.

(b) (4 points) Describe the two types of attractors that we have studied so far in LS 30.
5. (a) (4 points) State the definition of a linear function.

(b) (3 points) Suppose you have a function \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), and you know that
\[
f \left( \begin{bmatrix} 12 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \text{and} \quad f \left( \begin{bmatrix} -4 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.
\]
Could \( f \) be linear? Why or why not?

(c) (4 points) The function \( g: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is linear, and
\[
g \left( \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad g \left( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}.
\]
Since \( \begin{bmatrix} -2 \\ 22 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \), what is \( g \left( \begin{bmatrix} -2 \\ 22 \end{bmatrix} \right) \)?
6. Consider the following system of differential equations:

\[
\begin{cases}
P'(t) &= 0.9R(t - 0.4) - 0.2P(t) \\
Q'(t) &= 1.2P(t - 5) + \frac{S(t)^{12}}{1+S(t)^{12}} - 0.3Q(t) \\
R'(t) &= \frac{9}{9+P(t)^2} - 0.05R(t) \\
S'(t) &= 3 \cdot \frac{Q(t-8)^{10}}{1+Q(t-8)^{10}} - 0.1S(t)
\end{cases}
\]

(a) (4 points) Draw a feedback diagram (i.e., cause/effect diagram) for this system.

(b) (4 points) Would you expect this model to have oscillatory behavior? Why or why not? **Be as specific as possible!**

Question 6 continues on the next page...
(c) (4 points) Suppose that you simulate this model, and it turns out that it does not have any oscillatory behavior, but you want some variables to oscillate. What parameters could you change, and how would you change them, in order to make the system more likely to oscillate? **Again, be specific!**
7. Suppose $f: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear function for which

$$f \left( \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 9 \\ 3 \end{bmatrix}, \quad f \left( \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \quad \text{and} \quad f \left( \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

(a) (5 points) Compute $f \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$, $f \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$, and $f \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

(b) (5 points) What is $f \left( \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right)$?