Exercise 4.2.FE 1  Some people have difficulty maintaining a stable weight. Instead, they gain a lot of weight, go on a diet, lose the weight, but then gain it back. This pattern is sometimes referred to as yo-yo dieting.

a) What kind of feedback loop is involved in this situation?

b) Use your understanding of feedback loops and oscillations to suggest what might help such a person to stabilize their weight.

Part a:
Negative Feedback Loop (since a gain in weight will ultimately leads to the decrease in weight, and in reverse)

Part b:
The oscillations are caused by:

1. High sensitivity (steep reaction by the person): too much weight → drastic diet → too low weight → eat a lot → too much weight → repeat

2. Time delay by the body because it takes time for body weight to change

It is not as feasible to control the time delay since it is a physiological process. Thus, the person can only stabilize their weight by not changing their diet too drastically. Just adjust a little. Let the weight stabilize around a new value. If that is still not the desired weight, then make additional small change to the diet.

Exercise 4.2.FE 2  While traveling, you find yourself in a hotel room in which using the thermostat leads to large oscillations in the room’s temperature. The thermostat responds to the room’s air temperature by turning on an air conditioner on the other side of the room if the temperature near the thermostat gets too warm. Similarly, when the temperature near the thermostat gets cold, the air conditioner switches off. What could the builder of the hotel have done to prevent the oscillations you are experiencing?

Oscillations are caused by:

1. Steep reaction (high sensitivity)

2. Time delay

In this case, the reaction is steep (on/off) but it is not controllable.

Thus, we can only control the time delay.

To reduce drastic oscillations, we need to reduce time delay. In this case, it can be argued that the time delay is at maximum because the distance between the air conditioner and the thermostat is the greatest. Thus, the distance the air has to travel is also the greatest.

To reduce time delay, they can put the thermostat closer to the AC (or AC closer to the thermostat, either way works).
Exercise 4.2.FE 4  Meerkats are highly social small carnivores that live in southern Africa. They rely on each other to raise their young. Use the following assumptions to model the number of adult meerkats, M, in a population. You can invent parameters as necessary.

- The per capita rate at which meerkats give birth to babies who survive to adulthood is a steep sigmoid function of the adult population, with higher reproductive success at higher populations.
- Meerkats die of natural causes at a constant per capita rate d.
- Meerkats are preyed upon by eagles and jackals. These predators have many other prey, so their population does not depend on the meerkat population.
- The rate at which jackals prey on meerkats is a nonsigmoid saturating function of the meerkat population.
- The rate at which eagles prey on meerkats is a sigmoid function of the meerkat population. The sigmoid is not very steep.

**Bullet #1:**
For this one, it is up to interpretation, so this answer is acceptable:

\[
\frac{M'}{M} = k \cdot \frac{M^n}{(a_1)^n + M^n} \rightarrow M' = k \cdot \frac{M^n}{(a_1)^n + M^n} \cdot M
\]

However, one can argue that the number of adults now is dependent on the number of babies born \(\tau\) years back. And we know that the number of babies born is dependent on the number of adults \(AT\ THAT\ TIME,\ not\ at\ this\ moment.\ As\ a\ result,

\[
\frac{M'}{M(t - \tau)} = k \cdot \frac{[M(t - \tau)]^n}{(a_1)^n + [M(t - \tau)]^n} \rightarrow M' = k \cdot \frac{[M(t - \tau)]^n}{(a_1)^n + [M(t - \tau)]^n} \cdot M(t - \tau)
\]

**Bullet #2:**

\[
\frac{M'}{M} = -d \rightarrow M' = -d \cdot M
\]

**Bullet #4:**

\[
M' = -\frac{c_1 \cdot M}{a_2 + M} \cdot J
\]

**Bullet #5:**

\[
M' = -\frac{c_2 \cdot M^s}{(a_3)^s + M^s} \cdot E \quad (s \ text{ is \ small})
\]
Putting things together:

\[
M' = k \cdot \frac{[M(t-\tau)]^n}{(a_1)^n + [M(t-\tau)]^n} \cdot M(t-\tau) - d \cdot M(t) - \frac{c_1 \cdot M(t)}{a_2 + M(t)} \cdot J(t) - \frac{c_2 \cdot [M(t)]^s}{(a_3)^s + [M(t)]^s} \cdot E(t)
\]

Exercise 4.2.FE 5  The garibaldi is a large orange fish that lives off the coast of California and Baja California. Use the assumptions below to write a differential equation for the size of an adult garibaldi population.

- The number of adults joining a population is the number of eggs laid times the fraction that hatch times the fraction that survive to adulthood.
- Garibaldis lay eggs at a constant per capita rate, b.
- Because garibaldis sometimes eat their own eggs, the fraction of eggs that hatch is a declining sigmoid function of the adult population.
- Larval garibaldis float as plankton before becoming adults and joining a population. Thus, the number of individuals joining a population is proportional to the number that hatched six years earlier, with proportionality constant r.
- Adult garibaldis die at a constant per capita rate d.

\[
G' = \frac{\text{ (# of eggs)}}{b \cdot G(t-6)} \cdot \left( \frac{1}{k \cdot a^n + [G(t-6)]^n} \right) \cdot \frac{\text{ (fraction hatching)}}{r} \cdot \frac{\text{ (fraction survive)}}{d \cdot G(t)}
\]

Exercise 4.3.1  Explain what each term in this model means and why it has the algebraic form (for example, \(SP^2\)) that it does.

- \(v_0\) - the production rate of fructose-6-phosphate (from the two-step conversion of glucose not featured in reaction scheme)

- \(SP^2\) - the probability that ONE molecule of fructose-6-phosphate and TWO molecules of ADP meet each other AT THE SAME TIME (\(P^2\) because there are two molecules of ADP)

- \(c\) - the probability constant, which signifies the success rate of converting from having three molecules meeting together to a successful formation of another ADP molecule
cSP^2 - the rate at which one molecule of fructose-6-phosphate is lost (and conversely, one molecule of ADP is formed)

k - the per-capita rate of degradation of ADP

Exercise 4.3.2 Why can h act as a half-saturation density? In other words, what is the consumption rate when N = h, and what does this mean biologically?

At N = h,

\[ f(N = h) = \frac{C_{\text{max}} \cdot h}{h + h} = \frac{C_{\text{max}} \cdot h}{2 \cdot h} = \frac{C_{\text{max}}}{2} \]

Thus, mathematically, we see that at N = h, the rate of consumption of the predator is at half of its maximum rate.

Biologically, it can act as a switch between low and high stable values:

- Above h, an increase in N will have less effect in the rate of consumption. Thus if N increases at a really fast rate, the predators cannot keep up, thus the number of preys increases to a higher value.

- Below h, an increase in N will cause a big increase in the rate of consumption. Thus if N increases even though for slightly, the predators can eat a lot, thus the number of preys decreases to a lower value.

Exercise 4.3.3 Find the equilibria for this model using the parameter values in Figure 4.34. (Hint: Work with the second equation first.)

We have the following equations:

\[ N' = r_1 \cdot N \cdot \left(1 - \frac{N}{k}\right) - \frac{w \cdot N}{d + N} \cdot P \]  

(1)

\[ P' = r_2 \cdot P \cdot \left(1 - \frac{j \cdot P}{N}\right) \]  

(2)

Let’s simplify equation (1) a little more, by factoring out the N. As a result, we will have:

\[ N' = r_1 \cdot N \cdot \left(1 - \frac{N}{k}\right) - \frac{w \cdot N}{d + N} \cdot P = r_1 \cdot N \cdot \left(1 - \frac{N}{k}\right) - \frac{w}{d + N} \cdot N \cdot P = N \left[ r_1 \left(1 - \frac{N}{k}\right) - \frac{w}{d + N} \right] \]  

(3)

For \( N' = 0 \):
From (3), we have:

\[ N = 0 \]  

(4)

or

\[ \mathbf{r}_1 \cdot \left(1 - \frac{N}{k}\right) - \frac{w}{d+N} \cdot P = 0 \]

\[ \rightarrow \mathbf{r}_1 \cdot (k - N) \cdot (d + N) - (w \cdot k) \cdot P = 0 \]

\[ \rightarrow P = \frac{\mathbf{r}_1 \cdot (k - N) \cdot (d + N)}{w \cdot k} \]  

(5)

For \( P' = 0 \):

From (2), we have:

\[ \mathbf{r}_2 \cdot P = 0 \]

\[ \rightarrow P = 0 \]  

(6)

or

\[ 1 - \frac{j \cdot P}{N} = 0 \]

\[ \rightarrow N = j \cdot P \]

\[ \rightarrow P = \frac{1}{j} \cdot N \]  

(7)

We know have four cases to work with where we will pick:

1. Equation (4) or (5), AND
2. Equation (6) or (7)

Case #1: Equation (4) and Equation (6)

This case is straightforward. We have: \( N = 0 \) and \( P = 0 \)

Case #2: Equation (4) and Equation (7)
We have: $N = 0$

As a result, according to equation (7),

$$P = \frac{1}{j} \cdot N = \frac{1}{j} \cdot 0 = 0$$

Therefore, our equilibrium point is the same as in the case #1: $N = 0$ and $P = 0$

Case #3: Equation (5) and Equation (6)

We have: $P = 0$

As a result, according to equation (7),

$$0 = \frac{r_1}{w} \cdot k \cdot (k - N) \cdot (d + N)$$

$$\rightarrow N = k = 7 \text{ or } N = -d = -1$$

Since $-1 < 0$, $N = -1$ is not within our state space.

As a result, our equilibrium point will be: $N = 7$ and $P = 0$

Case #4: Equation (5) and Equation (7)

We have:

$$P = \frac{r_1}{w \cdot k} \cdot (k - N) \cdot (d + N)$$

$$P = \frac{1}{j} \cdot N$$

We can set them to equal each other, to arrive at:

$$\frac{r_1}{w \cdot k} \cdot (k - N) \cdot (d + N) = \frac{1}{j} \cdot N$$

(8)

Let’s substitute values in. We have: $r_1 = 1$, $r_2 = 0.1 = \frac{1}{10}$, $k = 7$, $d = 1$, $j = 1$, and $w = 0.3 = \frac{3}{10}$.

As a result, our equation (8) becomes:

$$\frac{1}{3} \cdot \frac{10}{7} \cdot (7 - N) \cdot (1 + N) = \frac{1}{1} \cdot N$$

$$\rightarrow \frac{10}{21} \cdot (7 - N) \cdot (1 + N) = N$$
\[
\frac{10}{3} + \frac{60}{21} \cdot N - \frac{10}{21} \cdot N^2 = N
\]
\[
\frac{10}{21} \cdot N^2 + \left(1 - \frac{60}{21}\right) \cdot N - \frac{10}{3} = 0
\]
\[
\frac{10}{21} \cdot N^2 - \frac{39}{21} \cdot N - \frac{10}{3} = 0
\]
\[
10 \cdot N^2 - 39 \cdot N - 70 = 0
\]

Using the quadratic formula, we have:

\[
N = \frac{-(-39) \pm \sqrt{(-39)^2 - 4 \cdot 10 \cdot (-70)}}{2 \cdot 10} = \frac{39 \pm \sqrt{4321}}{20}
\]

\[
N \approx 5.2367 \text{ or } N \approx -1.3367
\]

Since \(-1.3367 < 0\), \(N = -1.3367\) is not within our state space.

As a result, \(N \approx 5.2367\)

Substituting back to equation (7), we have:

\[
P = \frac{1}{j} \cdot N = \frac{1}{1} \cdot N = N \approx 5.2367
\]

Therefore, our equilibrium point will be: \(N = 5.2367\) and \(P = 5.2367\) (you can round down to whole values if you want)

**In Summary:**

Our equilibrium points are:

- \(N = 0\) and \(P = 0\) (cases #1 and #2)
- \(N = 7\) and \(P = 0\) (case #3)
- \(N = 5.2367\) and \(P = 5.2367\) (case #4)

**Exercise 4.3.FE 1** Briefly explain the statement due to W. Smith, “Puberty is a Hopf bifurcation.” What does this mean?

It means that when the person is really young, certain hormones in the body remain at stable levels. When puberty hits, those levels are no longer stable at set values. Instead, the concentrations of hormones will display oscillating behavior as a function of time.
Exercise 4.3.FE 6  Recall the Higgins-Selkov model of glycolysis,

\[
S' = V_0 - cSP^2 \\
P' = cSP^2 - kP
\]

a) Simulate this model with \(V_0 = 0.5\), \(c = 0.23\), and \(k = 0.4\) for three different initial conditions. How does the system behave?

b) In real life, for these parameter values, \(V_0\) can range from 0.48 to 0.6. Using any method you choose, approximate the value of \(V_0\) at which the system begins to have persistent oscillations. (You may want to use more than one method.)

Part a:

The trajectory, vector field, and corresponding time series are shown below:
As time goes by, the system approaches a closed loop (i.e. the limit cycle attractor), where the $S$ and $P$ values will oscillate as a function of time between set values.

**Part b:**
Any value between 0.52 and 0.53 is acceptable for this problem.