1. Prove that for any integer \( n \), \( 1 | n \) and \( -1 | n \).

2. For what integers \( n \) is it true that \( 0 | n \)? Prove your answer.

3. For what integers \( n \) is it true that \( n | 0 \)? Prove your answer.

4. Why is \( \text{gcd}(0,0) \) not defined? \( (\text{Hint: One of the previous two questions answers this.}) \)

5. Apply the Division-with-Remainder Theorem (a.k.a. the division algorithm) to each of the following pairs of numbers \( a, b \). That is, for each one, find integers \( q \) (quotient) and \( r \) (remainder) such that

\[
a = b \cdot q + r \quad \text{with} \quad 0 \leq r < b.
\]

(a) \( a = 47, b = 13 \)
(b) \( a = 823, b = 48 \)
(c) \( a = -79, b = 17 \)
(d) \( a = -6257, b = 316 \)
(e) \( a = 39582723, b = 8243 \)
(f) \( a = 82373852, b = 29574 \)

Note: You could do all of these by hand using good old fashioned long division. However, for some of these, it will likely be much faster to use a calculator, and I will allow you to use a (non-graphing, non-programmable, non-cellphone) calculator on the quizzes and the final exam for similar types of computations. A simple pocket calculator should suffice. If you haven’t tried it before, see if you can figure out an efficient way to use such a calculator for this type of computation.

6. In the original statement of the Division-with-Remainder Theorem, we assumed that \( a \) and \( b \) are integers with \( b > 0 \). In this problem you will come up with a more general version of this theorem.

(a) Suppose that \( a \) and \( b \) are integers, but with \( b < 0 \). Let \( b' = -b \), and apply the Division-with-Remainder Theorem to \( a \) and \( b' \). Rewrite the result in terms of \( a \) and \( b \).

(b) Verify that with your result from part (a), when \( b \) is negative, it is still true that there exist integers \( q \) and \( r \) such that \( a = b \cdot q + r \), but now the restriction on \( r \) can be stated as

\[0 \leq r < -b\]
Note that in general, whether \( b \) is positive or negative, the condition on \( r \) can be stated as

\[
0 \leq r < |b|
\]

(c) Prove the uniqueness of \( q \) and \( r \) in this more general version of the theorem. You should be able to do this by just verifying that all steps in the original uniqueness proof still work.

7. Apply the theorem just proved in the previous problem to each of the following pairs of numbers \( a, b \). That is, for each one, find integers \( q \) (quotient) and \( r \) (remainder) such that

\[
a = b \cdot q + r \quad \text{with} \quad 0 \leq r < |b|.
\]

(a) \( a = 47, b = -13 \)
(b) \( a = 956, b = -27 \)
(c) \( a = 29657452, b = -4382 \)

8. (a) Prove that for any natural numbers \( a \) and \( b \), \( 2^a - 1 \) divides \( 2^{ab} - 1 \).
(b) Use the result from part (a) to prove that for any natural number \( n \), if \( 2^n - 1 \) is prime, then \( n \) must be prime.
(c) Is the converse of part (b) true? That is, if \( n \) is prime, does it have to be true that \( 2^n - 1 \) is prime?